TRANSACTIONS OF
THE
OPTICAL SOCIETY

VOL. XX. DECEMBER, 1918 No. 3.

# SPIRIT LEVELS

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*Read and discussed, 12th December, 1918*

IT is a regrettable fact that the best tubes for spirit levels are made in Germany (mostly by Pessler of Saxony). The difference between a good and an indifferent tube probably depends on small details. A knowledge of these details does not seem to exist in this country, and I venture to bring the matter up in the hope that discussion here, and further investigation, may enable the British made bubbles to be improved. My remarks apply especially to accurate levels; there is actually not so much difficulty in obtaining bubble tubes for comparatively rough work, i.e. tubes that will indicate the horizontal with an accuracy of the order of a few minutes of arc.

Spirit levels are used with many instruments, rough and precise, for the purpose of determining the horizontal, or measuring *small* angles from the horizontal.

A spirit level consists of a glass tube shaped so that the longitudinal section of its upper internal surface is an arc of a circle. The tube is nearly filled with liquid, so that a small air space is left. This air bubble rises to the highest portion of the tube, and the line joining the two ends of the bubble is horizontal. The glass tube is generally fixed in some metal mount. If it is then placed on a horizontal surface and the position of rest of the bubble marked on the tube, the bubble will always return to this position when the mount rests on a horizontal surface. The horizontality of the surface can always be tested by reversing the bubble tube end for end,—if the surface is truly horizontal the bubble will come to rest at the same point in the tube in both positions of the mount. By adjusting the tube in its mount this position can be made central in the tube.

If the tube is slightly tilted the bubble will move, and the distance it will move in the tube depends on the amount of the tilt, and the radius of curvature of the tube. If $R$ is the radius of curvature of the tube, and the bubble runs a distance $S$

along the tube, the angle of tilt in circular measure will be $S/R$. Hence, if we measure the displacement of the bubble due to tilting the tube through a small angle, and know the radius of curvature of the tube, we can ascertain the angle through which it has been tilted.

If the level is to be used in this way it is obviously necessary for the curvature to be uniform.

If the bubble tube be mounted, as on a theodolite, so that it is attached to a frame which is capable of rotating about a vertical axis, the bubble will come to rest in the same position in its tube through whatever angle the frame is rotated, provided the axis is truly vertical.

The bubble tube is generally marked with equal divisions on its upper surface.

The sensitiveness of a bubble tube is generally measured by the angle of tilt necessary to move the bubble through one division of the tube. The linear value of one division is generally (for sensitive bubbles) 2 mm., $\frac{1}{10}$ inch (2·54 mm.), or 1 Paris line (2·26 mm.). Smaller bubbles are frequently divided to half this value.

It will be seen later that the accuracy obtained when using a bubble depends on other factors than its "sensitiveness." The sensitiveness is however very important, and should be marked on all bubble tubes. I have heard of complaints that (e.g.) an engineer's level *A* would not keep its adjustment so well as another maker's *B* which "never got out of adjustment." The instrument *B* proved to have a fine large bubble with a very low sensitiveness so that the instrument might be considerably out of adjustment without the bubble showing it. The mere fact that the bubble does not move when the instrument is reversed or rotated does not show that it is absolutely level, but only that the dislevelment is not sufficient to show on that particular bubble tube. The sensitiveness must be selected with due regard to the accuracy required.

I regret I know very little about the manufacture of bubble tubes, but I understand that the earlier bubble tubes were merely tubes slightly bent, and polished inside by means of a burnisher on a piece of wire pulled backwards and forwards by hand. It required much selection and testing to obtain a good tube.

Later tubes are ground out on a machine designed for the purpose. These are often ground barrel shaped, so that the tube is symmetrical about its axis, and such a tube should be capable of being used on its side or upside down. Even now, however, I fancy that such tubes are generally tested and the surface which proves most satisfactory is marked for use as the top of the tube.

As the length of the air bubble is important, and as it changes with temperature, sensitive tubes are sometimes made with a partition near the end; a small hole at the bottom of the partition allows liquid to be passed from one side to the other by suitably tilting the tube, thus adjusting the length of the air bubble. Such tubes are said to be chambered.

I am enabled, by the courtesy of Messrs E. R. Watts and Son, to show some tubes of various makes to-night. It will be noticed that the Pessler tubes look much cleaner, and are all marked with the angular value of one division. My experience is that the marked value is very nearly correct, and the curvature of

each tube is uniform. The British tubes I have tested vary considerably from their marked value and the sensitiveness varies considerably along the scale. I have not been able to make very exhaustive tests, so I do not give figures.

The requirements of a good spirit level are:

(i) The bubble should rapidly and accurately take up its correct position in the tube. This means as long an air bubble, and as short a radius of curvature, as possible.

(ii) Any tilt of the tube larger than the permissible error (for the class of work required) should cause the bubble to move a noticeable amount.

The greater the radius of curvature the greater the movement of the bubble for a given tilt, but the slower the movement. If a bubble is made too "sensitive" the movement will become unduly slow. Owing to this the accuracy of reading does not increase directly with the radius of curvature. Hence the smaller the movement of the bubble that can be detected, the shorter the radius of curvature that can be used, and the more satisfactory the bubble will be. This is purely a question of the method of viewing and illuminating the bubble.

(iii) The curvature should be uniform so that a given tilting of the tube should produce the same displacement of the bubble over the graduated portion of the tube.

When the tube is tilted through a small angle and the ends of the bubble cease to be at the same level, a force due to this difference of level tends to move the liquid. This force is proportional to the specific gravity of the liquid and to the difference in level of the ends of the bubble. The difference in level or "head" is $h$ where $h = l \sin \alpha$, $l$ being the length of the bubble and $\alpha$ the angle of tilt. We see therefore that this force varies directly as the length of the bubble and is independent of the curvature.

This force has to overcome the inertia of the liquid to be moved, and the friction. The less the distance the bubble has to move to assume its new position, the less the work necessary to move the liquid, and the more rapid the movement. Hence the speed of the movement increases with reduced sensitiveness. If there were no friction, or damping, the bubble would oscillate indefinitely about its new position —some friction is therefore necessary.

These frictional forces will depend on the nature of the liquid, of the glass, and the character of the glass surface. I am unable to say what is the best liquid or glass, or state of polish (or otherwise) of the surface. It seems obvious that the liquid should have a high specific gravity, and a low viscosity, and also a low surface tension. The surface tension effects depend not only on the liquid and its purity but also on the vapour of which the air bubble is composed and on the state of the surface, cleanliness, and polish. These points seem important, but I regret I can throw little light on them; their importance may be illustrated by taking an extreme case. If the liquid were water, and there were a patch of grease on the glass, it will be seen at once that when the end of the bubble reaches this patch considerable force would be needed for this liquid to move on to the patch, and the movement of the bubble would become very irregular, quite apart from any

actual irregularity in the curvature of the tube. I hope that the discussion, or subsequent research, will give further information on these points.

*Irregularity of surface*

The accuracy required in forming the surface of the tube is very great. If the grinding tool vibrates as it moves along the tube it may produce a corrugated surface. For the sake of a numerical illustration I will assume that the actual surface is a sine curve $\left(\text{equation } y = k \sin \frac{2\pi x}{p}\right)$ superimposed on the true circle; where $y$ is the linear departure from the true curve at any point whose distance along the curve is $x$, $k$ is the maximum value of $y$, and $p$ the period of the sine curve.

For a displacement $dx$ of the bubble the change of level of one end of the bubble is $\frac{l\,dx}{2R} + (y_1 - y_2)\cos\theta = h$, where $l$ is length of bubble, $R$ the radius of curvature, and $\theta$ the angle between the radius at the end of the bubble and the vertical; thus $\theta = \frac{l}{2R}$, and cos $R$ is so nearly 1 that it can be neglected.

The head produced is $h_1 + h_2$ where $h_1$ and $h_2$ are the changes at each end; the angular tilt corresponding is $\frac{h_1 + h_2}{l} = \alpha$. Hence

$$\alpha = \frac{k}{l}\left\{\sin\frac{2\pi x}{p} - \sin\frac{2\pi(x+dx)}{p} - \sin\frac{2\pi(x+l)}{p} + \sin\frac{2\pi(x+l+dx)}{p}\right\} + \frac{dx}{R}$$

$$= \frac{2k}{l}\sin\frac{\pi l}{p}\left\{\cos\frac{2\pi x + \pi l}{p} - \cos\frac{2\pi x + \pi l + 2\pi dx}{p}\right\} + \frac{dx}{R}$$

$$= \frac{4k}{l}\sin\frac{\pi l}{p}\sin\frac{\pi dx}{p}\sin\frac{\pi(2x+l+dx)}{p} + \frac{dx}{R}.$$

The first term represents the error in sensitiveness at the point considered.

We see that in general the variation is reduced by increase in $l$ but the term $\sin\frac{\pi l}{p}$ will vary from 1 to 0 according as $l$ is varied. If $l = np$, $\sin\frac{\pi l}{p} = 0$ where $n$ is any integer; if $l = (n + \frac{1}{2})p$, $\sin\frac{\pi l}{p} = \pm 1$.

Applying this to a bubble with radius of curvature 20 metres (sensitiveness 10″·3 per mm.), and an irregularity represented by $k = \cdot0005$ mm., $p = 10$ mm., length of bubble 35 mm., and taking $dx$ as 1 mm., we get that the tilt to move the bubble 1 mm. varies from 6″·8 to 13″·8, or a variation of ± 38 per cent.

If the bubble is increased in length to 40 mm. the error vanishes.

If we take $R$ as 100 metres (sensitiveness 2″·06 per mm.) and $l = 65$ we find that the angle of tilt necessary to move the bubble 1 mm. varies from 0″·9 to 3″·9, in other words it will be very difficult to set the bubble level, and no accuracy can be obtained in reading small angles of dislevelment. These figures show that the tube must be very accurately shaped.

Fig. 1 illustrates the case where $l = 4{\cdot}5p$; six different positions (11, 22, etc.) of the bubble have been shown, equally spaced along the tube; it is seen at once

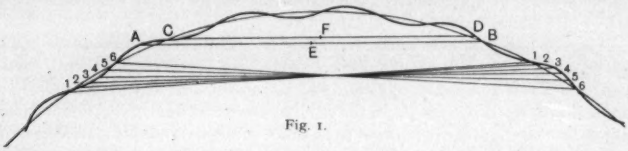



Fig. 1.

that the angles between successive positions are by no means uniform. Fig. 2 illustrates the case where $l = 5p$; in this case increased steepness of the glass surface at one end of the bubble is exactly balanced by reduced steepness at the other, and the angle for a given displacement remains constant. These figures also illustrate the effect of change of adjustment with change of bubble length. Suppose the bubble $AB$ (Fig. 1) were lying evenly between the marks on the tube, if the bubble shortens to $CD$ but without the tube being tilted, then $AC$ is considerably greater than $BD$, and the centre of the bubble has moved from $E$ to $F$ thus altering its adjustment. It would appear impossible to calibrate a tube, as the errors vary as the bubble alters its length.

How far the above really represents the type of irregularities met with I do not know, but it is not uncommon to find a bubble that will move much more readily in one direction than in the other (from a certain point) or one that is difficult to start moving and when started is difficult to stop—this is specially noticeable with short bubbles, and sensitive bubbles, as is to be expected.

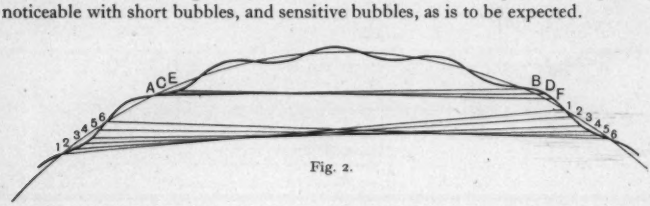



Fig. 2.

This is shown in Fig. 2 if $CD$ is the position to which it is desired to set the bubble; moving it from a position $AB$, a small tilt will move it to $CD$, but to move it on to $EF$ will require a considerable tilt; the bubble stops naturally at $CD$. Moving it from $EF$ a considerable tilt is required to bring it to $CD$, and the least bit too much sends it on towards $AB$. It is difficult to set it.

To reduce these defects it is necessary to reduce $k/l$; lengthening $p$ will also reduce the effects to some extent.

It is obvious from the above that it is desirable to use as long a bubble as possible, and with as short a radius of curvature as possible. The better the method of viewing, the shorter the radius of curvature for a given accuracy.

The general method of mounting the bubble is to fix it in a metal tube, a portion of which is cut away so that the graduated portion of the top of the bubble tube can be seen. The appearance of the end of the bubble varies with the direction from which the light falls.

There is also a parallax effect depending on the thickness of the glass—this is specially noticeable with a long bubble. The bubble is also frequently mounted in such a position that it cannot be viewed except from the side. One undesirable result of this is that in many cases (as in the ordinary engineer telescopic level) the bubble cannot be viewed from the position from which the observer uses the telescope—he has to walk around the instrument to view the bubble. This movement may tilt the whole instrument unless the ground on which the instrument stands is very firm.

A better means of illuminating the bubble would seem to be by transmitted light, the mounting tube being cut away below as well as above, and a suitable reflector placed below. I believe blackening the sides of the tube will increase the definiteness of the end of the bubble.

Some of the disadvantages are overcome by the use of a mirror above the bubble tube, so that it can be viewed from one end. This enables the observer to see the bubble without moving from the telescope, but it does not overcome parallax, and, as (except for one particular position of the eye, for each position of the mirror) the two ends of the bubble are at different distances from the eye, the apparent size of the divisions on the tube will not be the same at the two ends. Hence an error is liable to be introduced when setting the bubble evenly between marks.

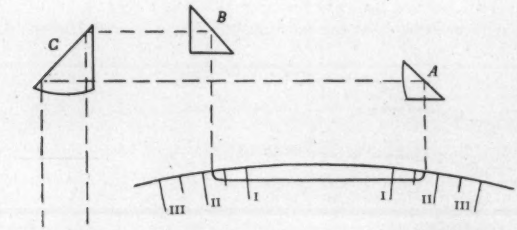



Fig. 3

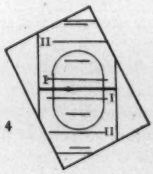



Fig. 4

An arrangement of prisms illustrated in Fig. 3 is used in the French geodetic level, and a somewhat similar one in the level designed by the Coast and Geodetic

Survey of the U.S.A. Prisms *A* and *B* are movable longitudinally to allow for the variations in length of the bubble; *C* and *D* are turned out of the plane of the paper so as to bring the eyepiece into a convenient position. The bubble is seen in the eyepiece considerably shortened, as shown in Fig. 4; this facilitates setting. By suitably curving the face of prism *A* the apparent size of the divisions at both ends can be made equal. By suitably curving one face of *C*, or by inserting a spectacle lens, the bubble can be focussed when the length of the rays from the eye to the bubble is less than the distance of distinct vision. Lenses could be introduced, if desired, to give an enlarged image of the bubble.

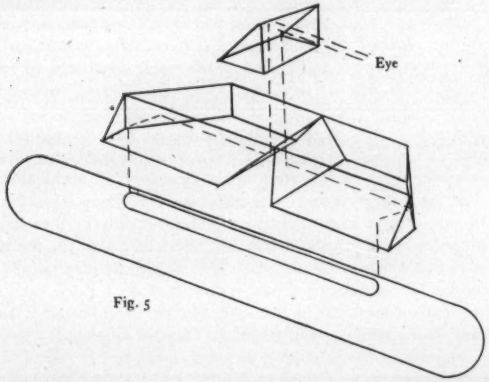



Fig. 5

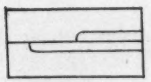


Fig. 6

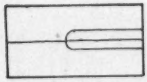


Fig. 7

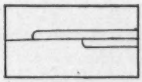


Fig. 8

A somewhat different arrangement is adopted by Messrs Zeiss; this is illustrated in Fig. 5, and is the nicest I have seen. One half of each end of the bubble is seen, as shown in Figs. 6 to 8. As the bubble moves, the two ends move apparently in opposite directions, and the bubble is set by bringing the two ends into coincidence, as shown in Fig. 7. This more than doubles the accuracy of setting, as the separation of the two ends as seen in the prism is double the actual movement of the bubble causing the separation. It is also much easier to bring two lines into coincidence, than to place a bubble evenly between marks. Another advantage is that the final adjustment of the bubble is done by moving the whole prism arrangement, instead of tilting the bubble tube; this is a far coarser and therefore easier mechanical adjustment, for a given angular adjustment. This method gives great accuracy of setting with a short radius of curvature tube. Zeiss bubbles that I have worked

with, having a sensitiveness of about 5″ per mm. (radius of curvature about 40 metres), can be set easily with a probable error of $\frac{1}{8}$ second, and with care to about half this.

I have seen an attempt to get the ease of adjustment of the Zeiss bubble mounting by putting the scale on an outer movable tube instead of on the bubble tube. This method increases the parallax errors, and is hardly to be recommended. Doubtless other methods can be devised.

*Fixing the glass tube in its mount.* With less sensitive bubbles the glass tube is generally set in its mount by plaster of Paris. With more sensitive ones various methods are adopted, such as a 3-point support at each end, the top support at each end being lined with cork, and pressed down with a spring. Whatever the method of mounting adopted it should be such that no strain is put on the glass at any temperature at which it is to be used. Such strain would not only alter the curvature, but might also alter the character of the glass surface and so increase the friction.

*Effects of temperature.* The most noticeable effect of rise of temperature is to expand the liquid, and so shorten the bubble. If a sensitive bubble is required for use where large variations of temperature are to be expected, a chambered tube is almost essential for accurate work. The importance of fixing so that no strain is brought on the glass by change of temperature has already been explained. If the glass be unequally heated, the tube will be distorted. From the figures given above it is obvious that a very small distortion from this cause may be very noticeable. The Zeiss bubble mentioned above is protected by a thick tube of glass surrounding the metal mount of the bubble tube, and the outer surface of the metal is polished. This should considerably minimise unequal heating.

Another small point in the mounting is worth noticing. If the bubble with its mount be fixed over or under the telescope of (e.g.) an engineer's level, and the metal arms supporting it get unequally heated from any cause (such as more sun on one side than the other) the bubble will get out of adjustment with the telescope. If on the other hand the bubble be fixed at the side of the telescope, such unequal heating will not alter the adjustment.

The following points appear to me to be worth further investigation:

(1) Quality, polish, and cleanliness of glass.

(2) Quality and purity of the liquid.

As regards these points the degree of polish may considerably affect the friction. Slight lack of cleanliness of the glass surface may very considerably affect the surface tension. Similarly very small impurities in the liquid, or vapour, may largely affect the surface tension.

The liquid should apparently have a high specific gravity and low surface tension, and low viscosity, and preferably a low temperature coefficient.

Any chemical action between the glass and liquid will almost certainly be detrimental, both by impairing the polish of the glass and by contaminating the liquid. Some sorts of glass are slightly soluble in water; if such glass be used, and water be present in the liquid, the bubble will gradually deteriorate.

The liquid should not alter its composition under the action of sunlight.

Liquids frequently used are absolute alcohol, sulphuric ether, and chloroform; the last named alters its condition under the influence of sunlight, and the bubble has a tendency to grow.

(3) The most suitable relation between the length of bubble and diameter of tube. This will to some extent affect the damping, and also the shape of the air bubble.

(4) The most suitable methods of mounting, viewing, and illuminating the bubble. These have been considered at some length above.

Lack of time has prevented me from making any exhaustive search into the literature of the subject, but I have noted the following references:

1. General.
   - (i) *Encyclopédie des Travaux Publics*, "Lever des Plans et Nivellement: 3$^{e}$ Partie, Nivellement de Haute Précision," by Ch. Lallemand. Paris (1889). [There is also a later edition.]
   - (ii) "Recherches sur les Instruments, les Méthodes, et le Dessin Topographiques," A. Laussedat. Paris (1898). [Historical.]
   - (iii) *Handbuch der Vermessungskunde*, W. Jordan. Stuttgart (1908). [7th revised edition.]
   - (iv) *Zeitschrift f. Instrumentenkunde*, 10 (1890), 309, 347.
   - (v) *Zeitschrift f. Vermessungswesen* (1891), 277.
   - (vi) Liagre, "Sur les oscillations du niveau à bulle d'air et sur les moyens d'y remédier," Grunert's *Archiv des Math. u. Phys.*, 6 Teil (1845), 404, and 7 Teil (1846), 1.
2. Investigation into chemical effects (solubility of glass, etc.).
   - (i) Hensoldt, "Das orthoskopische Ocular, von Karl Kellner, nebst einem Anhang zur Kenntniss und genauen Prüfung der Libellen," von M. Hensoldt, Mechaniker, Braunschweig (1849).
   - (ii) Rieth, "Unregelmässigkeiten der Libellen," *Zentralzeitung f. Optik und Mech.* (1889), 135; *Zeitschrift f. Vermessungswesen*, 16 (1887), 297. "Einfluss des Glases auf die Beweglichkeit der Libellen," Dingler's *Polytechnisches Journal*, 264 (1887), 501.
   - (iii) Weber, "Ursachen der Fehler an Libellen," *Zentralzeitung f. Optik und Mech.* (1888), 253.
   - (iv) Mylius, "Über die Störungen der Libellen," Mittheilung aus der physikalisch-technischen Reichsanstalt, *Zeitschr. f. Instrumentenkunde*, 8 (1888), 267, 428. Mylius u. Förster, "Über die Löslichkeit der Kali- und Natrongläser in Wasser," *Zeitschr. f. Instrumentenkunde*, 9 (1889), 117–122. Löwenherz, "Störungen bei Libellen," *Zeitschr. f. Instrumentenkunde*, 11 (1891), 166.
     See also *Zeitschr. f. Instrumentenkunde*, 12 (1892), 170.
3. Description of Zeiss level.
   *Transactions of the Institution of Mining Engineers*, 46, Part 2, 285–310.

The above references to German publications have been taken from Jordan's Handbook (1908); probably further information will be found in the later numbers of the *Zeitschrift für Instrumentenkunde*, and the *Zeitschrift für Vermessungswesen*.

## DISCUSSION

Dr R. S. Clay described an experiment he had carried out some years ago. He fixed a straight tube on Vs resting on a board which could be tilted suddenly through a definite angle of variable magnitude. A number of different liquids were tested, and he had found that a bubble in liquid $SO_2$ travelled very much more rapidly than in any of the other liquids tried. The rapidity of travel varied parabolically with the amount of slope, and for a given amount of tilt it increased with the length of the tube and, within limits, with the length of the bubble. In choosing a suitable liquid three factors should be considered: (1) viscosity, (2) difference in density of liquid and vapour, and (3) vapour tension.

Mr E. H. Rayner contributed the following remarks:

The production of accurate and sensitive levels is a matter of really great importance. Our knowledge of geophysics, a subject which is receiving more and more attention, is largely dependent for its progress on levelling operations of the greatest possible accuracy. Our knowledge of the rise and fall of the earth's surface is dependent on the spirit level.

A few figures will help to give some idea as to the dimensions of the quantities that have to be dealt with.

One second corresponds to a slope of 1 part in 206,000 which is equivalent to 4·8 millimetres in a kilometre or 0·31 inch in a mile. As the bubble can be easily read to $\frac{1}{8}$ of a millimetre we can therefore read to about 0·1 inch in a mile. Sensitive levels have commonly a movement of 1 millimetre, corresponding to a tilt of 1 second. Such a level would have a radius of curvature of 200 metres. Major Henrici informs me that the standard of levelling to be obtained, according to international agreements, is to be such that the probable error should not exceed in millimetres the figure obtained by taking the square root of the number of kilometres over which the levelling is taken. Over a distance of 25 kilometres the probable error should therefore not exceed 5 millimetres. To look at it in a larger way, if one levelled round the world, the probable error ought not to exceed $\sqrt{45,000}$ or 210 millimetres. This means that it would be an even chance whether you were more or less than 210 mm. out. Actually on first class survey work this accuracy is considerably surpassed, and the main error appears to be not in the want of accuracy in the instruments but in the movements of the ground.

As I trust some steps may be taken to improve the accuracy of bubbles made in this country, it may be that a statement of what appear to be the physical qualities of importance will be useful.

It is interesting to note that in all the sensitive bubbles which I have seen the glass appears never to have been polished. It is used as it comes off the grinding machine, and it may well be that polishing is more likely to do harm than good. This is of some interest, as the rugosities of the ground glass must be considerably larger in dimensions than the permissible inequalities of a bubble. It appears that it is only their small size, and the averaging of them, assisted by the

effect of surface tension, that produces the effect of continuity. The roughness is very largely masked by the similarity of the refractive indices of the glass and liquid. It can be seen if the finger is placed on the glass above the air bubble, the film of liquid will evaporate with the rise of temperature, and in a few seconds the inside surface becomes dry, when a liquid of low boiling point is used. The ground glass effect at once appears. But in ordinary use this drying does not take place, showing that we have to do with a complete liquid film surrounding the air bubble. The bubble probably depends for its satisfactory operation on the continuity of this film, which quite possibly is more easily maintained, in the case of large air bubbles, when the glass has a matt surface than when it is optically polished.

To turn now to the physical properties of the liquid which are of importance, I think that mobility probably comes first. Mobility may be taken to be the reciprocal of viscosity. The value of a really sensitive bubble depends primarily on the quickness with which it comes to rest. Ordinary commercial bubbles, relatively insensitive, give no trouble in this respect.

As a good basis for a list of liquids in order of decreasing viscosity we may take glycerine. The comparison shows that it is about 1000 times as viscous as the commonly used liquids.

The values given in the table are in C.G.S. units.

| Substance | Viscosity | Boiling Point Cent. | Density | Surface Tension |
|---|---|---|---|---|
| Glycerine ... ... | 10·0 | 290 | 1·3 | 65 |
| Mercury ... ... | 0·2 | 357 | 13·6 | 540 |
| Alcohol ... ... | 0·012 | 78 | 0·80 | 25 |
| Water ... ... | 0·010 | 100 | 1·00 | 80 |
| Carbon tetrachloride | 0·010 | 76 | 1·58 | 25 |
| Toluene ... ... | 0·006 | 111 | 0·88 | 25 |
| Chloroform ... ... | 0·0057 | 61 | 1·52 | 30 |
| Ether ... ... | 0·0034 | 35 | 0·73 | 17 |
| Pentane ... ... | 0·0026 | 37 | 0·62 | — |
| Hexane ... ... | 0·0035 | 64 | 0·65 | 17 |

From the above table it will be seen that where pure ether is not permissible, in the tropics for instance, the use of toluene or carbon tetrachloride might be worth considering. Pentane, which boils a couple of degrees above ether, has a viscosity considerably less, and appears to be well worth experimenting with, as, if it has no disadvantages, it ought to prove the better in this respect.

Other things being equal one would expect a high density to be useful in assisting a bubble to come to a definite position of rest. Here of course mercury would have a great advantage if it could only be persuaded to "wet" glass as the others do. We must take it as nature has given it to us, and be content with other liquids.

The heaviest are carbon tetrachloride and chloroform. The latter appears to be ruled out by its slow molecular change; but most likely the former is stable and its density is a good point in its favour in addition to its high boiling point.

In considering the density it appears that the rate of change of density with temperature may be a serious cause of unknown errors in bubbles. Suppose we

have a tube 10 mm. bore and the air bubble is 50 mm. long. A tilt of 1 second would correspond to a difference in level of 50/206,000 or 0·00024 mm. between the ends of the bubble. Now suppose one end of the bubble is slightly warmer than the other by reason of the sunlight or the presence of the observer, the liquid at the warmer end becomes lighter and takes more room causing the end of the bubble to move. Taking the depth to be 10 mm. we find that the increase in depth of 0·00024 mm. will be produced by 0·02° C. rise of temperature in the case of such a liquid as chloroform which has a coefficient of expansion of 0·0012. We see therefore how important evenness of temperature is and that surveying apparatus of precision might well be painted white rather than black much more than is commonly the case.

Another very important point is what may be termed the "bubbleness" of the bubble depending upon its surface tension. There is no doubt that a good demarcation line is wanted, which depends on the surface tension. Whether the surface tension should be any greater than is necessary to give this effect is difficult to say. From the table it will be seen that mercury is quite unique in this respect. Everyone knows how a small globule of mercury "sits up" in spite of its great density. The value for water is probably in practice much less than that given which refers to newly formed surfaces. It is much affected by contamination.

The rate of change of surface tension with temperature may possibly be the most important temperature effect. It is a general rule that the surface tension diminishes with rise of temperature, and this rate of diminution is the important matter. The result is that the surface tension of the colder and stronger end overcomes to some extent the opposition of the weaker which gives way and the bubble moves towards the warmer end.

From general thermodynamic considerations there would also appear to be an effect due to the absolute value of the surface tension.

As the bubble moves along the tube when coming to rest there is fresh air-liquid surface formed at one end and annulled at the other. The result is that there must be surface temperature effects producing a tendency to resist such alteration, the amount of which is proportional to the absolute value of the surface tension. The result in practice would be a thermodynamic sluggishness of the bubble in coming to its final position of rest. Whether this effect is of any importance it is difficult to say. One important effect of the absolute value of surface tension is that if it is small as in such liquids as sulphur dioxide, mentioned by Dr Clay, which is above its atmospheric boiling point, the "bubbleness" will be relatively small and the interface comparatively flat. The gas space would be shallow, and one would not expect satisfactory stability of position or ease of reading.

It is interesting to compare the surface tension energy with the energy of restoring motion of gravity when the bubble is slightly displaced. Take for example a bubble 5 cm. long and 1 cm. wide and imagine it to be displaced by an amount corresponding to 1 second. We have then the equivalent of a wedge shaped volume of fluid 2·5 cm. long, 1 cm. wide and 2·5/206,000 thick at the base. This is the excess at one end of the bubble, and, to assume the position of equilibrium, it

virtually has to fill a corresponding wedge shaped hollow at the other. The mass of the wedge is $\frac{1}{2} \times 2{\cdot}5/206{,}000 \times 1{\cdot}0 = 6$ millionths of a gram assuming the density to be unity. The vertical movement of the centre of mass of this wedge for the one second alteration is $2 \times 2{\cdot}5/206{,}000 \times \frac{1}{3} = 8$ millionths of a centimetre. Taking the value of $g$ as approximately 1000 (instead of the usual 981) we have the energy represented in the movement to be

$$\frac{6 \times 8 \times 1000}{1^6 \times 10^{12}}$$

or 48 thousand-millionths of a dyne-centimetre. Now the surface tension of the usual liquids is about 30 which means that a bubble of the dimensions taken, having a surface of about 100 square cm., will have a surface energy of about 3000 dyne-centimetres. We see that the surface tension energy is something of the order of a million million times the energy of restoration for 1 second displacement of the bubble. This shows how important small changes in surface tension may be, such as would be caused by small differences in temperature.

The above considerations are put forward not as a result of experience, but in the hope that a contribution on certain physical properties of importance may be of some use to those who are interested in the subject.

Mr S. D. Chalmers thought it should be possible to test the inner surface of the level tube by means of a test plate method, making use of the interference fringes formed with monochromatic light.

In considering the question of suitable liquids it was necessary to choose one which showed well-marked boundaries at the bubble.

The President (Prof. Cheshire) said he was sorry he must agree with everything the author had said as to the superiority of German levels. There could be no doubt about it. When he was at the Ministry of Munitions great difficulty was experienced in obtaining supplies of the simplest level bubbles. A number of scientific men had been called in to assist, but they had all failed.

Major Henrici, in reply, thanked the speakers for their very valuable suggestions. He did not know what liquid was employed in the Zeiss level, but thought that the outside of the bubble tube was probably blackened, as the boundaries of the bubble were well marked. The designer of the level had told him that the great mobility obtained was due to the care taken to prevent any strain on the glass of the bubble tube due to variations in temperature.

# THE SPHERICAL ABERRATIONS OF A THIN LENS*

By Instructor-Commander T. Y. Baker, R.N., B.A.
and Major L. N. G. Filon, R.A.F., D.Sc., F.R.S.

*Read and discussed, 9th May, 1918*

*INTRODUCTORY.* When rays of light originate from an object point, real or virtual, on the axis of a thin lens they will after traversing the lens pass approximately through the image point, which can be found by means of the ordinary equation: $1/v = 1/u + 1/f$.

For a small object in a plane at right angles to the axis through the object point and of height $l$ measured from the axis, there will be a corresponding image of height $Ml$ in the image plane where $M = v/u = 1 - v/f$. These are the Gaussian equations for the determination of the size and position of any small object. They are not exactly satisfied, the equations being in general more and more erroneous as height of the object and the inclination of the image-forming rays become larger and larger. The defects of the image are well known in type and it is only one of them, that due to spherical aberration, that is being considered here. The quantity $M$, the linear magnification of the image, is a quantity that we shall find necessary to employ but it should be noted that $M$ is really, in view of the relationship $M = 1 - v/f$, a quantity directly connected with the *position* of the image plane. If the values of $M$ be plotted in their correct image positions they form a uniform scale along the axis, the zero occurring at the back focus, the value 1 at the lens, 2 at the front focus, etc. $M$ is in fact a slightly more convenient quantity to use for some purposes to express the position of the image relatively to the lens.

*Notation.* The notation adopted throughout is that described in pages 69, 70 of the later paper and is illustrated in Fig. 2 of that paper.

*Aberration of a single ray.* For a ray which initially passes through the object point at an inclination $\alpha_0$ with the axis the deviation which it receives at the lens will be such as to cause it to cross the axis ultimately not through the Gaussian image point but at a distance from it which is the longitudinal spherical aberration. The amount of this longitudinal spherical aberration can be expressed as

$$a\alpha_4^2 + b\alpha_4^4 + \ldots\ldots,$$

where $a$, $b$, etc. are constants depending upon the image position, the shape and thickness of the lens, and the refractive index of the glass.

* This paper was read before the Optical Society on 9th May 1918. The MS of the paper was subsequently lost and as no copy had been kept it has been found impossible to reproduce the paper exactly as it was written. The paper as printed here is much condensed, but it is hoped that the main outlines have been preserved. (T. Y. B.)

The first approximation to this is clearly $aa_4^2$ or more conveniently $at_4^2$ where $t_4 = \tan \alpha_4$. The value of $a$ can be deduced from Coddington's formula and is easily established as

$$a = \frac{f(1-M)^4}{2n}\left\{(n+2)\left[\frac{C}{F} - \frac{n+1}{n+2}\cdot\frac{1+M}{1-M}\right]^2 + \frac{n^3}{4(n-1)^2}\left[1 - \frac{(n-1)^2}{n(n+2)}\left(\frac{1+M}{1-M}\right)^2\right]\right\} \quad \ldots.(1).$$

The value of $b$ is much more complex and it should be noted that the value of the coefficient of the second term will vary when we replace the circular measure of the inclination by the sine or tangent of that angle.

According as we retain the first, first two, first three, etc. terms of the longitudinal spherical aberration so we get normally a closer and closer approximation to the exact value. If the first term only is considered the aberration is spoken of as the first order aberration and so on.

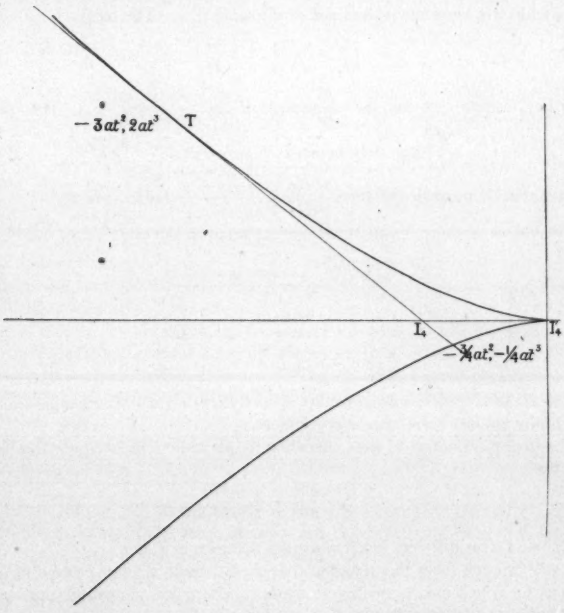



Fig. 1.

*First order Caustic.* If it be assumed that the longitudinal aberration is sufficiently closely expressed by the term $at_4^2$ for every ray which goes to the formation of the image point then it can be shewn that the rays all touch the semi-cubical parabola whose equation is $27ay^2 + 4x^3 = 0$, the origin of coordinates being the Gaussian image point and the positive direction of the coordinate $x$ being parallel to the optical axis, in the sense of travel of the light. The point of contact of the ray with the caustic has coordinates $-3at_4^2$, $2at_4^3$ and the ray cuts the caustic again at $-\frac{3}{4}at_4^2$, $-\frac{1}{4}at_4^3$, so that if $t_4$ is the slope of the extreme ray the radius of the *circle of least confusion* is $\frac{1}{4}at_4^3$.

If the value of $a$ is positive the caustic has its cusp pointing in the direction of the light as shewn in Fig. 1 and if negative against the light. The first is the more common case with a lens of positive power and the second with a lens of negative power, such lenses being then said to be under corrected.

If good image formation is desired it is clear that $a$ must be made as small as possible.

*Values of parameter of first order Caustic.* An inspection of formula 1 shews that $a$ and $f$ are more often than not of the same sign. The term

$$\left(\frac{C}{F} - \frac{n+1}{n+2} \cdot \frac{1+M}{1-M}\right)^2$$

must be positive but can be made zero if $\frac{C}{F} = \frac{n+1}{n+2} \cdot \frac{1+M}{1-M}$. The quantity $1 - \frac{(n-1)^2}{n(n+2)}\left(\frac{1+M}{1-M}\right)^2$ can only be negative provided $\left(\frac{1+M}{1-M}\right)^2 > \frac{n(n+2)}{(n-1)^2}$ which involves that $M$ must lie between $\frac{\sqrt{n(n+2)} - n + 1}{\sqrt{n(n+2)} + n - 1}$ and its inverse.

The values of these limits for $M$ are exhibited in the table below.

| $n$ | Limiting values of $M$. | |
|---|---|---|
| 1·50 | 0·643 | 1·559 |
| 1·55 | 0·621 | 1·613 |
| 1·60 | 0·600 | 1·667 |
| 1·65 | 0·581 | 1·720 |
| 1·70 | 0·564 | 1·777 |

Unless therefore the image position is such that the magnification falls between these limits no over correction is possible at all.

The greatest amount of over correction is possible if the shape of the lens is given by

$$\frac{C}{F} = \frac{n+1}{n+2} \cdot \frac{1+M}{1-M}.$$

The value of $a$ for different image positions is shewn in Fig. 2.

It will be seen from this diagram that there is only a limited range of image positions for which over correction of spherical aberration is possible and that the amount of over correction is minute.

*Limitation to the shape of the lens.* In order to obtain the maximum amount of over correction for any particular image positions or in order to reduce the under correction to a minimum we must put

$$\frac{C}{F} = \frac{n+1}{n+2} \cdot \frac{1+M}{1-M}.$$

For image positions a long way from the lens this tends towards a value

$$(n+1)/(n+2)$$

when the image is a long way beyond the lens and $-(n+1)/(n+2)$ when the image is a long way in front of the lens.

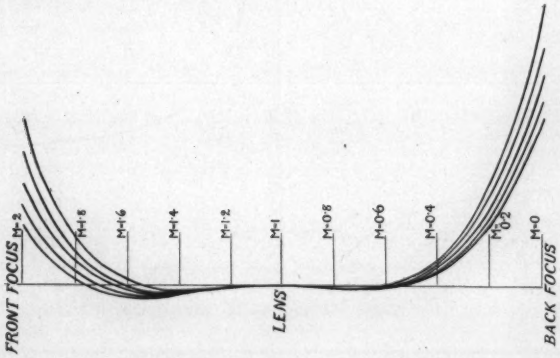



Fig. 2.

The actual value of $C/F$ is plotted in a graph (Fig. 3) for the value $n = 1{\cdot}5$. For the other refractive indices the curves are somewhat similar.

It will be seen that as the image position approaches the lens the shape of the latter must be more and more of the meniscus type in order to obtain over correction.

There is a physical limit beyond which we must not push the values of $C/F$ or we shall have a lens which is impossible to manufacture. The values $\pm 3$ have been assigned as the limiting values of $C/F$ for this purpose. The shapes of such a lens are shewn in Fig. 4 for the five refractive indices dealt with above.

Very roughly it will be seen that the range over which over correction is theoretically possible is the same as the range for which the numerical value of $C/F$ is greater than 3, and instead of its being possible to obtain over correction the limit which has to be put to the value of $C/F$ will introduce a positive term in $a$ where we were anticipating having a zero value. Figure 2 therefore requires modification to cover this case.

The extra positive term in $a$ is, with a lens of unit focal length,

$$(1-M)^2\frac{n+2}{2n}\left\{\frac{C}{F}(1-M)-\left(\frac{n+1}{n+2}\right)(1+M)\right\}^2.$$

This positive term more than counterbalances the negative terms except for a highly refractive glass within very narrow limits approximately $M = 1{\cdot}6$ to $M = 1{\cdot}75$.

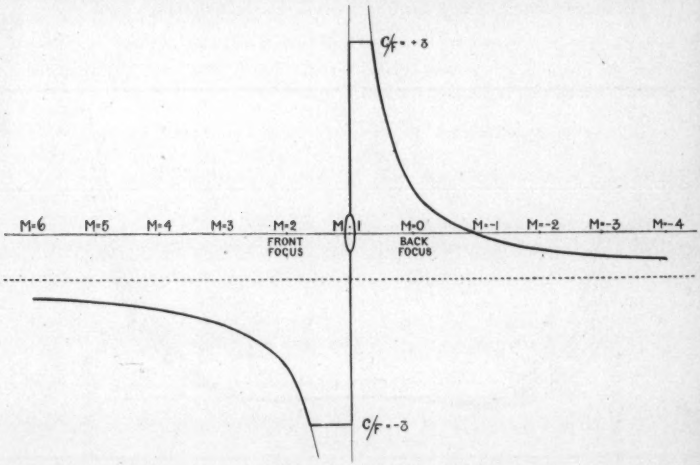



Fig. 3.

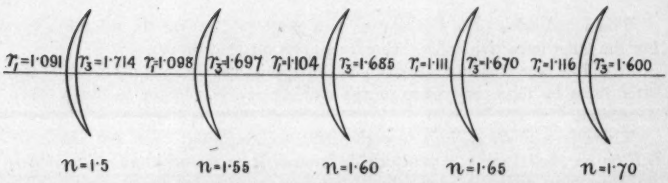



Fig. 4.

*Second order Caustic.* If the formula for the longitudinal aberration be more correctly represented by the expression $at^2 + bt^4$, where $t$ is the slope of the emergent ray, the semi-cubical parabola is only correct for the shape of the curve in the immediate neighbourhood of the image point. The equation of the ray in this case is $y + tx = -(at^3 + bt^5)$ and the point of contact is given by

$$x = -3at^2 - 5bt^4,$$
$$y = 2at^3 + 4bt^5.$$

There will be cusps in the curve where $\frac{dx}{dt} = \frac{dy}{dt} = 0$ or at places given by

$$6at + 20bt^3 = 0.$$

There are therefore three cusps upon the curve, viz. where $t = 0$ and where

$$t = \pm \sqrt{-\frac{3a}{10b}}.$$

The secondary cusps are real only when $a$ and $b$ are of opposite signs and their coordinates are $+\frac{9}{20} \cdot \frac{a^2}{b}$, $\pm \frac{6}{25} \cdot \frac{a}{b} \sqrt{-\frac{3a}{10b}}$. It will be seen that as $t$ is made to increase in size there is a possibility that the longitudinal spherical aberration will be first of all positive, if $a$ is positive, then go to a maximum and afterwards diminish to zero or below.

For reasons explained in the second paper it is not fair to expect that this second order caustic will necessarily be sufficiently exact as a representation of the curve for steep values of the inclination of the emergent ray.

## DISCUSSION

Mr H. S. Ryland thought the paper contained much that was of interest to designers. It would be useful if the permissible aberration limits for visual instruments could be determined. In all eyepieces, whether Huyghenian, Ramsden, or specially erecting, the results, after designing, could often be considerably improved by small alterations in the separations of the components. In general, the best definition over the whole field is obtained when the eyepiece is so adjusted that, on swinging the telescope, the field remains flat.

A Mr Zwillinger of Rochester, N.Y. had recently patented and put on the market an objective in which the last surface was aspherical. Using Hard Crown $n = 1{\cdot}5175$, $\nu = 57{\cdot}8$ and Dense Flint $n = 1{\cdot}6125$, $\nu = 36{\cdot}7$ in a two-piece cemented objective he was able to fulfil the sine condition, at the same time correcting spherical aberration and also colour for the $C$ and $F$ lines.

Mr T. Smith said he was much interested in the use of the caustic which the authors had described; their shapes were very interesting. He doubted however whether the secondary cusps were ever reached in some of the examples considered by the authors. If he remembered aright, Hamilton in his work on the characteristic function had determined the equations for caustics in terms of the characteristic function coefficients, but unfortunately had not expressed these in terms of the constructional data of the system.

He thought the considerable algebraic reduction referred to by the authors in obtaining the first order addition formula for spherical aberration quite unnecessary. The spherical aberration is determined for any system by some function $\alpha$

of its constructional data and the paraxial magnification where the equation to the caustic—apart from purely numerical factors—is $ay^2 = x^3$.

Suppose any two systems 1 and 2 are combined to form the system 12. The corresponding values of $a$ are $a_1$, $a_2$, $a_{1,2}$. Now $a_{1,2}$ must be of the form $pa_1 + qa_2$, for it is linear in the $a$'s and must vanish when $a_1$ and $a_2$ are both zero. Moreover $q$ must be unity, since $a_{1,2}$ and $a_2$ must be identical if $a$ is zero. To find $p$ we may take $a_2 = 0$. Under these conditions the 12 caustic will be the image of the 1 caustic according to Gaussian laws. In other words the equation

$$a_{1,2}y_{1,2}^2 = x_{1,2}^3$$

reduces on substituting $y_{1,2} = m_2 y_1$, $x_{1,2} = m_2^2 x_1$

to the equation $a_1 y_1^2 = x_1^3$.

It follows that in this special case

$$a_{1,2} = pa_1 = m_2^4 a_1,$$

and in the general case $a_{1,2} = m_2^4 a_1 + a_2$.

If $\beta$ denotes the coma coefficient, $\gamma$ the curvature or astigmatism constant, $\delta$ the distortion constant, and $\epsilon$ the constant for spherical aberration in the stop, which determines the variation in the distortion as the magnification changes, it is very easy on general principles to shew that the general laws for addition are

$$\begin{aligned}
a_{1,n} &= \Sigma a_\lambda m^4{}_{\lambda+1,n} \\
\beta_{1,n} &= \Sigma \beta_\lambda m^3{}_{\lambda+1,n} s_{\lambda+1,n} \\
\gamma_{1,n} &= \Sigma \gamma_\lambda m^2{}_{\lambda+1,n} s^2{}_{\lambda+1,n} \\
\delta_{1,n} &= \Sigma \delta_\lambda m_{\lambda+1,n} s^3{}_{\lambda+1,n} \\
\epsilon_{1,n} &= \Sigma \epsilon_\lambda s^4{}_{\lambda+1,n}
\end{aligned}$$

where each summation is taken from $\lambda = 1$ to $\lambda = n$, and $m_{\lambda+1,n}$, $s_{\lambda+1,n}$ are the magnifications which the image and the stop image subsequently undergo in the rest of the system.

The authors' formula for $a$ for a single thin lens was a particular case of a general formula he (Mr Smith) had published earlier for any thin system of any number of components. It might be of interest to quote these for all the aberrations given above, taking $\gamma$ to refer to the primary focal line. Omitting common numerical factors

$$\begin{aligned}
a &= f(1-m)^4 \{4C + 2\varpi + 1 - 4(B' - B)M + (2\varpi + 3)M^2\} \\
\beta &= f(1-m)^3(1-s)\{4C + 2\varpi + 1 - (B' - B)(3M + S) + \varpi M(M + S) + M(M + 2S)\} \\
\gamma &= f(1-m)^2(1-s)^2\{4C + 2\varpi + 1 - 2(B' - B)(M + S) \\
&\qquad + \tfrac{1}{3}\varpi(M^2 + 4MS + S^2) + S(2M + S)\} \\
\delta &= f(1-m)(1-s)^3\{4C + 2\varpi + 1 - (B' - B)(M + 3S) + \varpi S(M + S) + 3S^2\} \\
\epsilon &= f(1-s)^4\{4C + 2\varpi + 1 - 4(B' - B)S + (2\varpi + 3)S^2\}
\end{aligned}$$

where $M = \dfrac{1+m}{1-m}$ and $S = \dfrac{1+s}{1-s}$. The effect of bending the lens by adding the common curvature $r/f$ to each surface of a standard conformation is given by

$$\begin{aligned}
\varpi &= \varpi_0 \\
B' - B &= B_0' - B_0 + 2r(1 + \varpi) \\
C &= C_0 + r^2(1 + 2\varpi).
\end{aligned}$$

The standard conformation for a single lens is that in which the curvatures of the two surfaces are equal in magnitude and opposite in sign. If the refractive index of such a lens is $\mu$

$$\varpi = 1/\mu$$

$$B_0' - B_0 = 0$$

$$4C_0 + 2\varpi + 1 = \{\mu/(\mu - 1)\}^2.$$

In general $\varpi$ is the Petzval sum expressed as a ratio. If non-spherical surfaces are introduced $\varpi$ and $B' - B$ are unaltered, but $C$ is changed. For first order aberrations every surface is equivalent to the osculating ellipsoid of revolution. The above equations, together with

$$\varpi_{1,n}/f_{1,n} = \Sigma\, \varpi_\lambda/f_\lambda,$$

contain practically the whole useful theory of the aberrations of thin lenses. In actual lenses the aberrations of higher orders had to be considered, and formulae for these could be constructed in a workable shape.

With regard to the equation $h = H\sqrt{\frac{\beta}{\beta}}$ the whole question as to what was the best correction was a difficult one and could not be solved on paper. The effects of colour, the differences of aberration due to difference in wave-length, and the presence of aberration in the eye had all to be taken into account. This offered a fruitful field for work. It was desirable to find out not only what the eye likes best, but also the permissible limits of error in certain directions. This would enable us to effect satisfactory compromises and to say that we had reached a point where the subject might be left.

Mr S. D. Chalmers thought that Mr Smith in his recent paper had not discussed all the problems treated in the present paper. Mr Smith's method of dealing with the subject of aberrations was very useful, and it was a pity that it had not been more used. Regarding the desirability of dealing with the average of the second and third order aberrations he agreed with the authors', rather than with Mr Smith's method of treatment. In a number of cases the second order aberrations as calculated algebraically were not sufficient. It was better to include partially the third order terms than to ignore them entirely.

It was a matter of great importance to determine the limits within which corrections must be made. The great difficulty was the necessity of exploring a large field, and it would be a great advance if it were possible to eliminate certain portions of the field. Considerable loss of time could often be prevented by using analytical methods.

Replying to Mr T. Smith, Major Filon pointed out that there was really no essential difference, except that of mathematical presentation, between the study of the complex of rays and that of the characteristic function defining the surface to which these rays are normal, and the results should therefore be identical.

With regard to the higher order caustics, Mr Baker and himself had calculated trigonometrically the aberrations for three select rays at the back focus of seven lenses corresponding to $C/F = 0, \pm{\cdot}69, \pm 93$ and $\pm 1{\cdot}93$ respectively. The constants of an empirical caustic had been calculated from the outer and paraxial rays and the fit checked against the middle ray. Major Filon gave some figures shewing

that for lenses of moderate mean curvature the fit was quite good, but for $C/F = \pm 1{\cdot}93$, the divergences of the middle ray from the empirical caustic were becoming more serious.

With reference to the first order aberration (or, as he would prefer to call it, the *third* order aberration, since it is really derived from the third order terms in the sine expansions) he would like to point out that the part of the spherical aberration constant which gave most trouble with positive lenses was the positive irreducible part, which remained when the shape had been adjusted to the best possible. Now the contribution of this part of the formula to the spherical aberration constant after passing through any number of positive lenses could be put into the simple form

$$\Sigma \frac{fn^2}{8(n-1)^2}(M'-M)^4,$$

$f, n$ being the focal length and refractive index of any lens of the system and $M, M'$ the subsequent magnification of the images formed immediately before and immediately after passing through this particular lens.

Putting the matter in this form it was obvious why a reversing lens must necessarily contribute largely to the final aberration, for then the fourth power of the numerical *sum* of the magnifications appeared as a factor and the contribution of this lens was accordingly magnified.

# ON AN EMPIRICAL FORMULA FOR THE LONGITUDINAL SPHERICAL ABERRATIONS IN A THICK LENS

By Instructor-Commander T. Y. Baker, R.N., B.A. and Major L. N. G. Filon, R.A.F., D.Sc., F.R.S.

*Read and discussed, 12th December, 1918*

SUMMARY

A formula of the type $q = \frac{At_4'^2}{1 + Bt_4'^2}$ provides a rapid and practically accurate method of obtaining the longitudinal spherical aberration in a lens, without the use of the laborious trigonometrical calculations.

1. The usual method of treating the aberrations of a symmetrical optical system is one of successive approximation based upon developments in ascending powers of the inclinations of the ray to the axis of the optical system or of the sine or tangent of these inclinations.

The successive terms of the power series provide what are known as the aberrations of first, second, third and higher orders. The developments in question will differ, according as we develop the aberrations in terms of the initial or in terms of the final inclination of the ray; and again, the inclination being selected, the development will differ (except as regards the first order aberration) according as we take for argument the circular measure, sine, or tangent. In default of any definite agreement as to which of these should be taken as the argument of the series, the aberrations of the second and higher orders lack precise definition.

In what follows we shall restrict ourselves entirely to rays lying in an axial plane, and originating from a point on the axis, so that we are dealing exclusively with spherical aberrations.

The axis of $x$ will be taken along the axis of the optical system, in the direction of propagation of the light, the axis of $y$ perpendicular to this in the plane of the rays.

For many purposes it is convenient to take the tangent $t$ of the final inclination of the ray as the argument of the developments. If we call $q$ the longitudinal spherical aberration of the ray which has slope $t$ on emergence, we have

$$q = f(t) \qquad \ldots\ldots(1),$$

and the equation of the ray, referred to the geometrical image as origin, is

$$\begin{aligned} y + tx &= -tq \\ &= -tf(t) \end{aligned} \qquad \ldots\ldots(2).$$

The envelope of the line (2) is the caustic on emergence. If the form of the function $f$ be known the caustic is easily found. Conversely if the caustic be given, rays are constructed by drawing tangents to it and the function $f$ can be obtained.

2. If we develop $f(t)$ in ascending powers of $t$ it is found that

$$f(t) = at^2 + bt^4 + ct^6 + \ldots \qquad \ldots(3),$$

$a, b, c$ being quantities depending upon the curvatures of the surfaces and the object and image positions, i.e. the magnification $M$.

If we stop at the first term, which will be a sufficient approximation when $t$ is small, we get for the ray

$$y + tx = -at^3$$

and the coordinates of a point on the envelope are given by

$$\left.\begin{aligned} x &= -3at^2 \\ y &= +2at^3. \end{aligned}\right\}$$

Eliminating $t$ we obtain the semi-cubical parabola

$$27ay^2 + 4x^3 = 0 \qquad \ldots(4),$$

which is the well-known first approximation to the caustic.

3. A second approximation is obtained by taking the first two terms of (3). The ray is then

$$y + tx = -(at^3 + bt^5).$$

The coordinates of a point $(x, y)$ on the envelope are expressed in terms of $t$ by the equations

$$\left.\begin{aligned} -x &= 3at^2 + 5bt^4 \\ y &= 2at^3 + 4bt^5 \end{aligned}\right\} \qquad \ldots(5).$$

The $x, y$ equation obtained by eliminating $t$ is complicated and need not here be written down. We note that the caustics will have cusps where $\frac{dx}{dt} = \frac{dy}{dt} = 0$, i.e. where

$$6at + 20bt^3 = 0,$$

or

$$t = 0 \text{ (principal cusp)},$$

$$t = \pm\sqrt{\frac{-3a}{10b}} \text{ (secondary cusps)}.$$

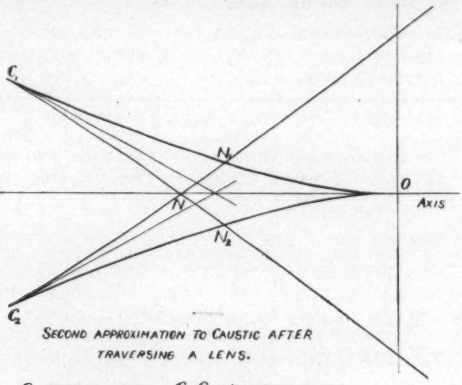
The latter are real only when $b$ and $a$ have opposite signs. The characteristics of the curve when the secondary cusps are real (which is the most frequent case) are shewn in Fig. 1. There is a node corresponding to $t^2 = -\frac{a}{2b}$.

4. So far as is known, only the constant $a$ has been calculated theoretically for a symmetrical optical system.

Its value can be deduced in the case of a thin lens from formula (1) Art. 146 of Herman's *Geometrical Optics*. A convenient formula for a thin lens has been given by the authors in the foregoing paper (p. 59). This is

$$a = \frac{f}{2n}(1-M)^4\left\{(n+2)\left[\frac{C}{F} - \frac{n+1}{n+2}\left(\frac{1+M}{1-M}\right)\right]^2 + \frac{n^3}{4(n-1)^2}\left[1 - \frac{(n-1)^2}{n(n+2)}\left(\frac{1+M}{1-M}\right)^2\right]\right\} \qquad \ldots(6)$$

where $f$, $F$, $n$, $M$ are the focal length, power, refractive index and magnification and $C = \frac{1}{2}\left(\frac{1}{r_1} + \frac{1}{r_3}\right)$ = mean curvature of lens (the radii $r_1$ and $r_3$ being measured towards the centres of curvature and reckoned positive in the direction of propagation of the light). In what follows comparisons will be made with the results




SECOND APPROXIMATION TO CAUSTIC AFTER TRAVERSING A LENS.

O PRINCIPAL CUSP; $C_1$, $C_2$ SECONDARY CUSPS.
N PRINCIPAL NODE; $N_1$, $N_2$ SECONDARY NODES.

Fig. 1.

of trigonometrical calculations of paths of rays through a set of lenses of finite thickness. It therefore becomes necessary to obtain a formula for $a$ in the case of a thick lens. No explicit formula of this kind is known to us. We proceed to calculate one.

5. The refracting media will be denoted by suffixes 0, 2, 4, the surfaces by suffixes 1, 3 (Fig. 2).

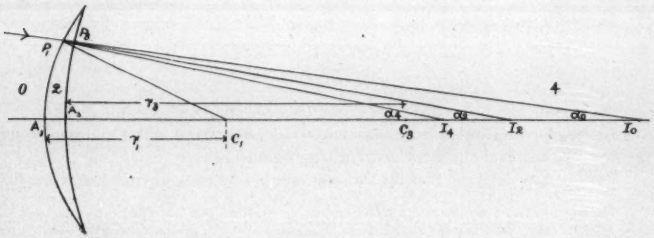



Fig. 2.

$C_1$, $C_3$ denote the centres of curvature of the two surfaces.

$A_1$, $A_3$ the points where the axis meets the surfaces.

$P_1$, $P_3$ the points where a given ray crosses these surfaces.

$\alpha_0$, $\alpha_2$, $\alpha_4$ the inclinations of the ray in media 0, 2, 4, reckoned positive when the ray converges towards the axis.

$I_0$, $I_2$, $I_4$ the points where the rays in media 0, 2, 4 meet the axis.

We will also write for shortness $t_0$, $t_2$, $t_4$ for $\tan\alpha_0$, $\tan\alpha_2$, $\tan\alpha_4$.

Suppose the rays $P_1I_0$, $P_1P_3I_2$, $P_3I_4$ to be affected by aberrations $-a_0t_0^2$, $-a_2t_2^2$, $-a_4t_4^2$*, then it is easy to shew that

$$a_2 = a_0\left(\frac{n_2}{n_0}\right)^3 M_{02}{}^4 + \tfrac{1}{2}M_{02}{}^4\frac{n_2}{n_0{}^3}(n_2-n_0)A_1I_0\left\{n_0\frac{A_1I_0}{r_1}-(n_0+n_2)\right\}\left\{\frac{A_1I_0}{r_1}-1\right\}^2 \quad ..(7)$$

where $M_{02}$ is the magnification produced by the first surface. This may be deduced from formula (A) p. 189 of Herman's *Geometrical Optics*, if we write in this formula

$$\alpha'/\alpha = (n_2/n_0)\,M_{02},\ \mu' = n_0,\ \mu = n_2,\ dv = a_0\alpha'^2,\ du = a_2\alpha^2$$

and use the equations

$$y = A_1I_0\alpha',\ n_2\alpha = \{n_0 + (n_2 - n_0)\,A_1I_0/r_1\}\,\alpha'$$

to transform the right-hand side.

Now, for a second refraction, taking the ray backwards

$$a_2 = a_4\left(\frac{n_2}{n_0}\right)^3 M_{42}{}^4 + \tfrac{1}{2}M_{42}{}^4\frac{n_2}{n_0{}^3}(n_2-n_0)A_3I_4\{n_0A_3I_4/r_3-(n_0+n_2)\}\{A_3I_4/r_3-1\}^2 \quad ..(8).$$

Eliminating $a_2$, and using $M_{02}/M_{42} = M_{04}$ = total magnification produced by the lens, we find

$$a_4 = a_0M_{04}{}^4 + \frac{n_2-n_0}{2n_2{}^2}[M_{04}{}^4A_1I_0\{n_0A_1I_0/r_1-(n_0+n_2)\}\{A_1I_0/r_1-1\}^2 - A_3I_4\{n_0A_3I_4/r_3-(n_0+n_2)\}\{A_3I_4/r_3-1\}^2],$$

or writing for shortness

$$n_0 = n_4 = 1,\ n_2 = n,\ M_{04} = M \text{ and } A_1I_0 = r_1 + C_1I_0,\ A_3I_4 = r_3 + C_3I_4,$$

$$a_4 = a_0M^4 + \frac{n-1}{2n^2}[M^4(C_1I_0+r_1)(C_1I_0/r_1-n)(C_1I_0/r_1)^2 - (C_3I_4+r_3)(C_3I_4/r_3-n)(C_3I_4/r_3)^2] \quad ....(9).$$

Now it is easy to shew that

$$C_1I_0/r_1 = f/Mr_1 - f/r_3 - n/(n-1)$$
$$C_3I_4/r_3 = -Mf/r_3 + f/r_1 - n/(n-1),$$

expressions in which the thickness of the lens does not appear explicitly.

In the above $I_0$ and $I_4$ have been treated as geometrical object and image, this being sufficient to the present order of approximation.

Putting now $a_0 = 0$ so that the incident rays belong to a punctual pencil, we find

* In the present paper longitudinal aberrations are reckoned positive when the ray crosses the axis *before* it meets the Gaussian image plane. This makes the longitudinal aberration positive for an under corrected convergent lens and this is the convention usually adopted by opticians.

$$a_4 = \frac{n-1}{2n^2}\left[r_1\left(\frac{f}{r_1}-\frac{fM}{r_3}-\frac{M}{n-1}\right)\left(\frac{f}{r_1}-\frac{Mf}{r_3}-\frac{Mn^2}{n-1}\right)\left(\frac{f}{r_1}-\frac{Mf}{r_3}-\frac{nM}{n-1}\right)^2\right.$$
$$\left.-r_3\left(\frac{f}{r_1}-\frac{fM}{r_3}-\frac{1}{n-1}\right)\left(\frac{f}{r_1}-\frac{Mf}{r_3}-\frac{n^2}{n-1}\right)\left(\frac{f}{r_1}-\frac{Mf}{r_3}-\frac{n}{n-1}\right)^2\right]$$
$$= \frac{n-1}{2n^2}\left[\frac{(r_1-r_3)f^4}{r_1^4}+\frac{(n+1)^2r_3f^3}{(n-1)r_1^3}-\frac{2n(1+n+n^2)r_3f^2}{(n-1)^2r_1^2}+\frac{n^2(n+1)^2r_3f}{(n-1)^3r_1}-\frac{n^4r_3}{(n-1)^4}\right.$$
$$+M\left\{-\frac{4(r_1-r_3)f^4}{r_1^3r_3}-\frac{4(n+1)^2f^3}{(n-1)r_1^2}+\frac{4n(1+n+n^2)f^2}{(n-1)^2r_1}-\frac{n^2(1+n)^2f}{(n-1)^3}\right\}$$
$$+M^2\left\{\frac{6(r_1-r_3)f^4}{r_1^2r_3^2}+\frac{6(n+1)^2f^3}{(n-1)r_1r_3}+\frac{2n(1+n+n^2)f^2}{(n-1)^2}\left(\frac{1}{r_1}-\frac{1}{r_3}\right)\right\}$$
$$+M^3\left\{-\frac{4(r_1-r_3)f^4}{r_1r_3^3}-\frac{4(n+1)^2f^3}{(n-1)r_3^2}-\frac{4n(1+n+n^2)f^2}{(n-1)^2r_3}-\frac{n^2(n+1)^2f}{(n-1)^3}\right\}$$
$$+M^4\left\{\frac{(r_1-r_3)f^4}{r_3^4}+\frac{(n+1)^2f^3r_1}{(n-1)r_3^3}+\frac{2n(1+n+n^2)fr_1}{(n-1)^2r_3^2}\right.$$
$$\left.\left.+\frac{n^2(n+1)^2fr_1}{(n-1)^3r_3}+\frac{n^4r_1}{(n-1)^4}\right\}\right] \quad \ldots.(10).$$

We notice from (10) that even when the thickness of the lens is taken into account, the aberration coefficient $a_4$ is still an expression of the fourth degree in the magnification but the coefficients can now no longer be expressed as rational functions of $C/F$. For we have, if $h$ is the thickness of the lens,

$$C=\frac{1}{2}\left(\frac{1}{r_1}+\frac{1}{r_3}\right), \quad F=(n-1)\left(\frac{1}{r_1}-\frac{1}{r_3}+\frac{n-1}{n}\frac{h}{r_1r_3}\right).$$

These equations lead to a quadratic for $\frac{1}{r_1}$ or $\frac{1}{r_3}$, and thus $\frac{1}{r_1}$ and $\frac{1}{r_3}$ are not expressible rationally in terms of $C$ and $F$.

If, however, we neglect squares of $h$, then we find

$$\frac{1}{r_1}=C+\frac{F}{2(n-1)}-\frac{h(n-1)}{2n}\left(C^2-\frac{F^2}{4(n-1)^2}\right),$$
$$\frac{1}{r_3}=C-\frac{F}{2(n-1)}+\frac{h(n-1)}{2n}\left(C^2-\frac{F^2}{4(n-1)^2}\right),$$

whence, after long and complicated reductions, we obtain

$$a_4=\frac{f}{2n}(1-M)^4\left[(n+2)\left\{\frac{C}{F}-\frac{1+Mn+1}{1-Mn+2}\right\}^2+\frac{1}{4}\frac{n^3}{(n-1)^2}-\frac{n^2}{4(n+2)}\left(\frac{1+M}{1-M}\right)^2\right]$$
$$+\frac{h(n-1)^2}{2n^3}\left[2(1-M)^4\left(\frac{C}{F}\right)^4-4\frac{n^2+2n-1}{n-1}(1+M)(1-M)^3\left(\frac{C}{F}\right)^3\right.$$
$$+(1-M)^2\left\{\frac{6n^2+5n-3}{n-1}(1+M)^2-\frac{4n(n+1)}{n-1}M\right\}\left(\frac{C}{F}\right)^2$$
$$+(1-M)(1+M)\left\{\frac{-2n^4+4n^3+n^2-2n+1}{(n-1)^3}(1+M^2)-\frac{2(n^2+2n-1)}{(n-1)^3}M\right\}\frac{C}{F}$$
$$-\frac{n^4-\frac{1}{2}n^3-\frac{1}{4}n^2-\frac{1}{8}}{(n-1)^4}(1+M^4)+\frac{n^3-n+\frac{1}{2}}{(n-1)^4}M(1+M^2)$$
$$\left.+\frac{n^3-\frac{1}{2}n^2-2n+\frac{3}{4}}{(n-1)^4}M^2\right] \ldots.(11).$$

6. The next proceeding was to calculate trigonometrically the aberrations for a number of lenses at the back focus, for three rays in each case, the outer one of which passed close to the edge of the lens; a formula of the type $at^2 + bt^4$ was then fitted to the longitudinal aberration, the coefficients $a$ and $b$ being calculated from the inner and outer rays and the accuracy of the fit was tested against the middle ray.

The object of this was twofold (1) to test how far the coefficient $a$, deduced from two oblique rays, would agree with the coefficient $a_4$, calculated above theoretically for paraxial rays, (2) to test how far a formula of the type $at^2 + bt^4$ could accurately represent the longitudinal aberrations and therefore whether it was worth while going through the very laborious calculation of the theoretical second order aberrations. For clearly if it could be shewn that a formula of the type $at^2 + bt^4$ was inadequate, the value of the inclusion of the second order terms was largely discounted.

The lenses selected were all positive lenses calculated to be of unit focal length and thickness = 0·0625, the diameter being about 0·5 in all cases. *The refractive index* was taken as that of *crown glass* = 1·52.

The radii and mean curvatures are given below.

| Description of Lens | $r_1$ | $r_2$ | $C/F$ |
|---|---|---|---|
| Meniscus ... ... | ·349420 | 1 | 1·930943 |
| Plano-convex ... | ·52 | ∞ | ·961539 |
| Crossed ... ... | ·603597 | −3·621582 | ·690306 |
| Equi-convex ... | 1·029197 | −1·029197 | 0 |
| Reversed crossed ... | 3·621582 | − ·603597 | − ·690306 |
| Reversed plano-convex | ∞ | − ·52 | − ·961539 |
| Reversed meniscus ... | −1 | − ·349420 | −1·930943 |

The three rays selected crossed the first principal plane at distances 0·075, 0·150, 0·225 from the axis. The results of the test are shewn in the table below. The fifth column shews the difference: True aberration of mean ray less aberration calculated from formula $at^2 + bt^4$.

| $C/F$ | $a$ (empirical) | $a$ (calculated) | Longitudinal aberration of mean ray | $\Delta q$ Error in aberration of mean ray |
|---|---|---|---|---|
| 1·930943 | 2·60819 | 2·63843 | +·060427 | −·002777 |
| ·961539 | 1·07275 | 1·07227 | +·024749 | −·000128 |
| ·690306 | ·99268 | ·99683 | +·022998 | −·000054 |
| 0 | 1·55876 | 1·56000 | +·036211 | −·000044 |
| − ·690306 | 3·25693 | 3·26845 | +·076413 | −·000654 |
| − ·961539 | 4·23844 | 4·27219 | +·100044 | −·001871 |
| −1·930943 | 9·00589 | 9·51627 | +·223242 | −·039045 |

Now, *on the whole*, the above results appear at first sight to give a fairly satisfactory answer to the two questions requiring solution. More particularly towards the centre of the range of $C/F$ the empirical $a$ agrees very tolerably with the calculated and also the middle ray agrees with the formula.

As we come to the extreme lenses of the set, however, serious divergences appear. For $C/F = -1{\cdot}930943$, the empirical $a$ differs seriously from the calculated and the error in the aberration of the mean ray, as calculated from the formula, is about $17\frac{1}{2}$ per cent.

7. Further numerical investigation confirmed the doubts as to the validity of the formula $at^2 + bt^4$ for the range of inclinations occurring here. A few examples worked out for a meniscus shaped lens of extreme curvature for which $C/F = 3$ revealed the fact that for such a lens the expression $at^2 + bt^4$ gave a highly unsatisfactory fit, the values of the constant $b$ calculated from the inner and outer rays, and from the inner and middle rays, being often widely different.

Further two lenses were examined for magnification $-1$. One was the plano-convex lens included in the previous series $C/F = 0{\cdot}961539$, the other was a lens of less mean curvature $r_1 = 0{\cdot}687946$, $r_3 = -2{\cdot}063838$, $C/F = 0{\cdot}484534$. In both these cases five rays were examined and the values of $\beta\,(\beta = q/t^2)$ were plotted to $t^2$. One of these is shewn in Fig. 3. If the formula $q = at^2 + bt^4$ were valid the graph of $\beta$ to $t^2$ would be a straight line. Instead of this it shews a very decided curvature. The other lens shewed similar characteristics.

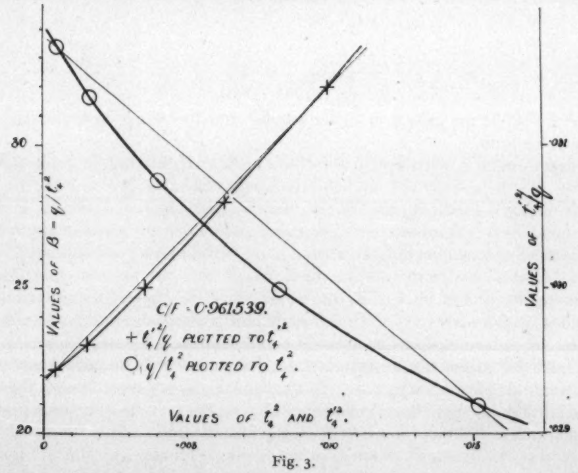



Fig. 3.

If we take the middle ray and the two extreme rays of the set corresponding to the graph of Fig. 3 and fit a formula

$$q = at_4^2 + bt_4^4 + ct_4^6$$

involving the next order of aberrations, we find that

$$a = 33{\cdot}9709, \quad b = -1413{\cdot}031, \quad c = 36421{\cdot}68.$$

We see therefore that the coefficients increase very largely in value as we take higher powers of $t_4$ so that for quite moderate values of $t$, the third term will more than balance the second.

8. This raises the whole question of the convergence of the power series for $f(t)$, a convergence which appears to have been generally assumed at any rate for ordinary inclinations. That this convergence must break down completely for certain regions is, however, apparent at once from the following considerations.

Let $I_0$ (Fig. 4) be the object point corresponding to a fairly large positive value of $M$ so that $I_0$ is slightly behind the front focus $F_0$.

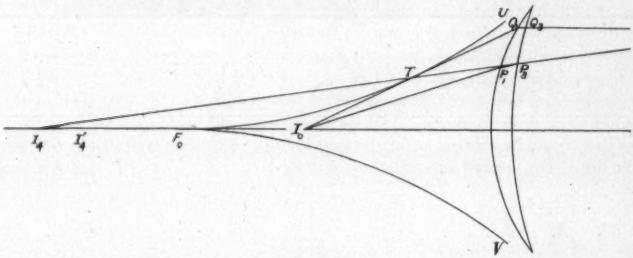



Fig. 4.

Let $UF_0V$ be the caustic at $F_0$ for parallel rays travelling backwards through the final medium.

Draw through $I_0$ a tangent $I_0T$ to the caustic $F_0V$, making an angle $\bar{\alpha}_0$ with the axis. By the property of reversibility of light, the ray $I_0T$ proceeding from $I_0$ will be refracted parallel to the axis, so that the corresponding $\alpha_4$ is zero, and the corresponding $I_4$ is at infinity. This fact alone shews that the series of aberrations of successive orders must fail, for when $t_4 = 0$ it should give $q = 0$ and not $q = \infty$.

But, further, the geometrical image $I'_4$ will in this case be some considerable (but finite) distance in front of $F_0$ (i.e. on the left in Fig. 4) and for *very small* inclinations we know the rays will necessarily have a small aberration. Thus a ray such as $I_0P_1$ will emerge as $I_4P_3$ and will have a finite $\alpha_4$.

It follows then that as $\alpha_0$ increases from 0 to $\bar{\alpha}_0$, $\alpha_4$ reaches a numerical maximum and then diminishes down to zero. (In the figure $\alpha_0$, $\alpha_4$ are negative with the convention of Fig. 2.) For higher inclinations than $\bar{\alpha}_0$, the emergent ray has a positive $\alpha_4$ and $I_4$ moves inwards from infinity in the final medium.

Fig. 5 shews the curve of $\tan \alpha_4$ to $\tan \alpha_0$ drawn for the lens $C/F = 1{\cdot}930943$ and $M = 2$. The maximum value occurs when $\alpha_0 = 13° 30'$. Also $\alpha_4 = 0$ when $\alpha_0 = 20°$ nearly. The greatest value of $\alpha_0$ for which a ray from the object will strike the lens is about 26°. It follows that for a large portion of the light passing through the lens we are well beyond both critical values.

Several important conclusions follow from the above as to the nature of the functions dealt with. In the first place it will appear that although $\alpha_4$ and the longi-

tudinal aberration $q$ must be (necessarily, since a ray is completely determined once its starting point and inclination are given) one-valued functions of $\alpha_0$, $a_0$ and $q$ are *not* one-valued functions of $\alpha_4$, for the same value of $\alpha_4$ must recur on both sides of the maximum, and this corresponds to different values of $\alpha_0$ and also to different values of $q$. Thus if $t = t_4$, $q = f(t_4)$ is a funetion of $t_4$ which is *at least* two-valued and has a branch point corresponding to $t_4 = \tan (\alpha_4)$, where $(\alpha_4)$ is the maximum value of $\alpha_4$.

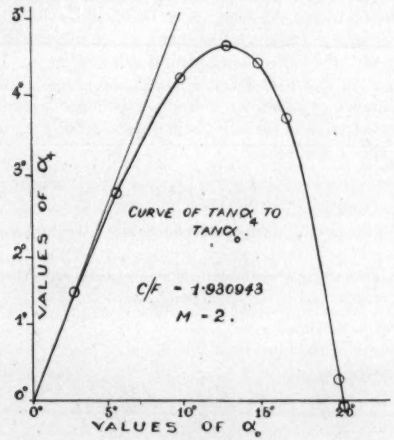



Fig. 5.

A development of $f(t_4)$ in a Taylor series round $t_4 = 0$—and this is all that the method of successive aberrations of increasing order is in effect—will, by a well-known result in theory of functions, be convergent only so long as $t_4$ is numerically less than mod $(\tan (\alpha_4))$ and this holds good even when the branch point becomes imaginary, which is the case when $I_0$ is to the left of $F_0$, so that $M$ is large and negative. Beyond such a value of $t_4$ the development fails *entirely*, however many orders of aberration we include. Thus in the case of the lens $C/F = 1{\cdot}930943$ and $M = 2$ the method of successive orders of aberration would fail, even if carried to an indefinite extent, to account for the behaviour of more than about $\frac{1}{4}$ of the light going through the lens.

Further, it is easy to see from Fig. 4 that the closer $I_0$ is taken to $F_0$, that is, the greater we take $M$ numerically, the smaller the value of $\bar{\alpha}_0$ and, consequently, of $(\alpha_4)$. In fact for large values of $M$ only an infinitesimal portion of the light passing through the system can be dealt with by the method of successive orders of aberration at all.

9. We see then that it is useless to take for $f(t_4)$ a one-valued function of any kind, if we are to get results valid over the whole range required.

This suggests that it would be preferable to take, instead of $t_4$, the tangent $t_0$ of the initial inclination of the ray, or the tangent $t'_4$ of the inclination of the "Gaussian" ray which goes through the geometrical image and which is the emergent ray as given by Gauss method. As $t_0 = Mt'_4$, either of these arguments is really equivalent to the other.

If we take now $q = f(t_0)$ it becomes possible to assume for $f$ a one-valued form, but this will be by no means expansible in a Taylor series over the whole range, for if $t_0 = \pm \tan \bar{\alpha}_0$, $\alpha_4 = 0$ and $q = \infty$, so that $t_0 = \pm \tan \bar{\alpha}_0$ are poles of $f$ and the radius of convergence of $f(t_0)$ is the numerical value of $\tan \bar{\alpha}_0$. Beyond this the Taylor series fails. So that here again, although the range is somewhat greater than before, the method of successive orders in $t_0$ is bound to come to grief.

Let $\tau_0$ = numerical value of $\tan \bar{\alpha}_0$. Then, if we suppose $f(t_0)$ to have no other poles within the range considered,

$$\left(1 - \frac{t_0^2}{\tau_0^2}\right) f(t_0)$$

will be regular over the range. We may develop this in a Taylor series and we reach the result

$$f(t_0) = \frac{a' t_0^2 + b' t_0^4 + c' t_0^6 + \ldots\ldots}{1 - \dfrac{t_0^2}{\tau_0^2}} \qquad \ldots\ldots(12)$$

as the development to be aimed at.

If the convergence of the numerator is sufficiently rapid*, we may stop at the first term and thus obtain

$$q = f(t_0) = \frac{a' t_0^2}{1 - \dfrac{t_0^2}{\tau_0^2}} = \frac{a' M^2 t'^2_4}{1 - \dfrac{M^2}{\tau_0^2} t'^2_4}$$

$$= \frac{A t'^2_4}{1 + B t'^2_4} \qquad \ldots\ldots(13).$$

It is important to note that the conclusions of this paragraph apply equally to a system of two or more lenses.

10. The next step has been to verify whether the numerical results already obtained for $M = 0$ and $M = -1$ could be adequately represented by a formula of the type (13).

To test the fit for $M = 0$, the two constant formula (13) was fitted to the inner and outer rays and the value of the longitudinal aberration $q$ calculated for the middle ray and compared with the actual $q$.

Also it appears, when we take $t'_4$ small, that the coefficient $A$ appearing in (13)

* At first sight it would appear that if this formula is correct the total aberration will vanish when $A$ vanishes, i.e. when the first order aberration is zero. At such a place the remaining terms in the numerator in (12) rise in relative importance but it should be noted that the aberration is very small in such cases. On the other hand where $A$ is large the importance of the denominator is proportionately great.

should be identical with the theoretical coefficient $a$ of the third order aberration. Forming a table as in par. 6 we obtain the following values:

| $C/F$ | $A$ | $a$ (theoretical) | $q$ for mean ray | $\Delta q$ |
|---|---|---|---|---|
| 1·930943 | 2·653174 | 2·638431 | ·060427 | −·000404 |
| ·961539 | 1·07447 | 1·072270 | ·024749 | −·000052 |
| ·690306 | ·996545 | ·996829 | ·022998 | −·000019 |
| 0 | 1·559347 | 1·559999 | ·036211 | −·000016 |
| −·690306 | 3·266938 | 3·268450 | ·076413 | −·000136 |
| −·961539 | 4·26761 | 4·272189 | ·100044 | −·000273 |
| −1·930943 | 9·45716 | 9·516270 | ·223242 | −·003191 |

A comparison of the above table with the one of par. 6 shews that not only has the new formula not worsened the fit, but it has considerably improved it. The divergences $\Delta q$ in the last column are considerably less than they were before in every case, and the constant $A$ also comes out closer to the theoretical $a$ than the empirical $a$, in every case but one. The exceptional case is one where the divergence is small anyhow. All the serious divergences have been improved.

Consider now the results for $M = -1$. Here five rays have been taken, which cross the first principal plane at distances from the axis of 0·04, 0·08, 0·12, 0·16, 0·20 respectively. If we use the extreme rays to determine the constants in each case, we find the divergences for the three middle rays to be as follows:

$C/F = 0{\cdot}484534$.

| $t'_4$ | $q$ | $\Delta q$ (old formula) | $\Delta q$ (new formula) |
|---|---|---|---|
| 0·04 | ·033906 | +·000284 | +·000012 |
| 0·06 | ·075836 | −·001450 | −·000003 |
| 0·08 | ·133740 | −·003030 | −·000048 |

$C/F = 0{\cdot}961539$.

| $t'_4$ | $q$ | $\Delta q$ (old formula) | $\Delta q$ (new formula) |
|---|---|---|---|
| 0·04 | ·054050 | −·001124 | +·000180 |
| 0·06 | ·119807 | −·006335 | +·000233 |
| 0·08 | ·208764 | −·014122 | +·000132 |

The fit, here again, is very much better. The crosses in Fig. 3 shew the values of $t'_4{}^2/q$ plotted to $t'_4{}^2$ and the straight line fitting them. This fit can be compared with the fit of $q/t_4{}^2$ plotted to $t_4{}^2$ on the same diagram. It seemed therefore reasonable to assume that a formula of type (13) will give a useful fit over a considerable range.

11. The next stage in the investigation has been to obtain systematically the values of the constants $A$ and $B$ for the five lenses $C/F = 0$, $\pm$ ·961539 and $\pm$ 1·930943, in each case for a number of image positions ranging from $M = 3$ to $M = -2$. In every case the two selected rays crossed the first principal plane at distances ·05 and ·15 from the axis. These were taken rather smaller than in the previous investigation, in the hope of getting a better determination of the constants

for moderate inclinations. In the case $M = 1$ this choice would have led to very high inclinations, and in the determination of the constant $B$ the value $M = 1$ was omitted. As, however, the total aberrations are extremely small for $M = 1$, any error in $B$ at this place will have little importance in practice. Values of $A$ for $M = 1$ were computed by taking rays through at a moderate inclination. $B$ however was then badly determined, as it was obtained from differences of nearly equal quantities. It should be noted that $M = 1$ is the case where the image and object are practically coincident at the lens, if the latter is thin.

12. We will first consider the values obtained for the coefficient $A$. These are tabulated below for each of the five lenses examined and for eleven values of $M$ in each case. In a parallel column have been entered the corresponding values of the coefficient $a$ computed from formula (10), and also the proportional differences $(A-a)/a$.

| | $C/F = 1{\cdot}930943$. | | | $C/F = {\cdot}961539$. | | |
|---|---|---|---|---|---|---|
| $M$ | $A$ | $a$ | $(A-a)/a$ | $A$ | $a$ | $(A-a)/a$ |
| 3·0 | 238·1715 | 238·2440 | −·000304 | 119·7148 | 119·7441 | −·000245 |
| 2·0 | 21·59256 | 21·5757 | +·000779 | 11·76524 | 11·76457 | +·000057 |
| 1·5 | + 2·433029 | + 2·42678 | +·002575 | + 1·46630 | + 1·46309 | +·002194 |
| 1·0 | − ·012978 | − ·013024 | −·003532 | − ·011627 | − ·011662 | +·003001 |
| 0·5 | + ·038636 | + ·038631 | +·001294 | + ·126740 | + ·126556 | +·001454 |
| 0·1 | 1·432817 | 1·43228 | +·000377 | ·641774 | ·641872 | −·000153 |
| 0 | 2·642022 | 2·63843 | +·001361 | 1·07308 | 1·072269 | +·000755 |
| −0·1 | 4·47233 | 4·47408 | −·000391 | 1·73724 | 1·738572 | −·000765 |
| −0·5 | 22·110000 | 22·11748 | −·000338 | 8·44280 | 8·42829 | +·001722 |
| −1·0 | 87·0549 | 87·0813 | −·000303 | 34·0414 | 34·2057 | −·004803 |
| −2·0 | 538·878 | 539·257 | −·000703 | 221·0908 | 221·1106 | −·000090 |

| | $C/F = 0$. | | | $C/F = -{\cdot}961539$. | | |
|---|---|---|---|---|---|---|
| $M$ | $A$ | $a$ | $(A-a)/a$ | $A$ | $a$ | $(A-a)/a$ |
| 3·0 | 48·9610 | 48·9715 | −·000216 | 16·7924 | 16·7870 | +·000322 |
| 2·0 | 5·53364 | 5·53388 | −·000004 | 2·02886 | 2·02489 | +·001961 |
| 1·5 | + ·825797 | + ·825775 | +·000027 | + ·419315 | + ·419390 | −·000179 |
| 1·0 | − ·011205 | − ·011219 | −·001248 | − ·011589 | − ·011662 | −·006260 |
| 0·5 | + ·345793 | + ·345868 | −·000217 | + ·734888 | + ·735285 | −·000540 |
| 0·1 | 1·18505 | 1·18616 | −·000936 | 3·20938 | 3·21020 | −·000255 |
| 0 | 1·55979 | 1·56000 | −·000135 | 4·27122 | 4·27219 | −·000227 |
| −0·1 | 2·04496 | 2·04075 | +·002063 | 5·57233 | 5·56945 | +·000517 |
| −0·5 | 5·62260 | 5·63414 | −·002048 | 13·8144 | 13·8194 | −·000362 |
| −1·0 | 16·8638 | 16·9112 | −·002803 | 34·2359 | 34·2057 | +·000883 |
| −2·0 | 90·2812 | 90·1462 | +·001498 | 134·864 | 134·853 | +·000082 |

$C/F = -1{\cdot}930943$.

| $M$ | $A$ | $a$ | $(A-a)/a$ |
|---|---|---|---|
| 3·0 | 16·8101 | 16·8567 | −·002759 |
| 2·0 | ·619692 | ·618090 | +·002592 |
| 1·5 | + ·175547 | + ·176159 | −·003474 |
| 1·0 | − ·013933 | − ·013024 | +·069794 |
| 0·5 | + 1·34614 | + 1·34848 | −·001735 |
| 0·1 | 6·94940 | 6·95297 | −·000513 |
| 0 | 9·51086 | 9·51627 | −·000568 |
| −0·1 | 12·6974 | 12·7077 | −·000811 |
| −0·5 | 33·6940 | 33·7036 | −·000285 |
| −1·0 | 87·0474 | 87·0813 | −·000389 |
| −2·0 | 353·870 | 353·880 | −·000028 |

Examination of the above tables shews that the agreement between the constant $A$ obtained empirically from two trigonometrically calculated rays and the constant $a$ theoretically calculated, is very close indeed in almost every case. If we leave out the values of the proportional error for $M = 1$, where the values of $a$ and $A$ are themselves so small that they exaggerate enormously the effect of quite negligible *absolute* errors upon the *proportional* error, the mean proportional error in the above table is ·000982 or the average divergence, in calculating the longitudinal aberration, using the theoretical constant $a$ instead of $A$, will be of the order of 1/1000th of the whole aberration. For practical purposes this accuracy should be ample. We therefore conclude that in formula (13) $A$ should be taken as identical with the constant $a$ given by formula (10).

13. We now have to examine the values of $B$. In this case there is no theoretical expression to guide us and it will be necessary to obtain an empirical formula giving as close a fit as we can obtain.

It is possible, however, to derive valuable indications as to the type of formula which we should attempt to fit, by considering the caustic at the front focus for parallel rays in the final medium, already discussed in par. 8. This caustic will be the caustic at the back focus for the *reversed lens*.

If we denote by $Q$ and $T_4$ the longitudinal aberration and slope on emergence of a ray which is parallel to the axis before approaching the reversed lens, we have

$$Q = f(T_4') = \frac{\mathscr{A}(T_4')^2}{1 + \mathscr{B}(T_4')^2} \qquad \ldots\ldots(14),$$

$\mathscr{A}$ and $\mathscr{B}$ being the constants $A$ and $B$ for the lens with the sign of $C/F$ reversed at $M=0$.

Now if the ray in question is the one passing through $I_0$ (Fig. 4), $Q = F_0 I_0$.

Hence
$$F_0 I_0 = \frac{\mathscr{A}(T_4')^2}{1 + \mathscr{B}(T_4')^2}.$$

Now *at the back focus* $T_4'$ never differs very seriously from $T_4$, for such inclinations as we are considering, and it is found that, by slightly altering the constants $\mathscr{A}$ and $\mathscr{B}$ to values $\mathscr{A}'$ and $\mathscr{B}'$ say, we can obtain a fairly accurate representation of the longitudinal aberration at the back focus from a formula

$$Q = \frac{\mathscr{A}' T_4^2}{1 + \mathscr{B}' T_4^2} \qquad \ldots\ldots(15),$$

and therefore
$$F_0 I_0 = \frac{\mathscr{A}' T_4^2}{1 + \mathscr{B}' T_4^2} \qquad \ldots\ldots(16),$$

where $T_4$ is the actual slope of the ray $I_0 Q_1$. But this $= \tan \bar{\alpha}_0 = \tau_0$ in the notation of pars. 8 and 9 (see p. 76). And from equation (13)

$$\tau_0^2 = -\frac{M^2}{B}.$$

Hence, substituting for $T_4^2$ the value $-\frac{M^2}{B}$ in (16) we obtain

$$F_0 I_0 = \frac{\mathscr{A}' M^2}{-B + \mathscr{B}' M^2},$$

or
$$B = \mathscr{B}' M^2 - \mathscr{A}' \frac{M^2}{F_0 I_0} \qquad \ldots\ldots(17).$$

But, by an elementary result

$$F_0 I_0 = \frac{f}{M} = \frac{1}{M},$$

since the focal length has been taken to be unity.

We have therefore

$$B = -\mathscr{A}' M^3 + \mathscr{B}' M^2 \qquad \ldots.(18),$$

giving a formula for $B$ in terms of the magnification $M$. It is clear that the formula (18) will only give a good approximation when $I_0$ is fairly close to $F_0$, that is, for numerically large values of $M$. For, as $I_0$ moves away from $F_0$, the slope $\tan \bar{\alpha}_0$ of $I_0 Q_1$ increases, that is $T_4$ in equation (16) increases and this equation ceases to provide an adequate approximation to $F_0 I_0$. The larger $M$, the closer the approximation.

This immediately suggests that $B$ should be fitted with a series in descending powers of $M$, of which the right-hand side of (18) gives the two leading terms.

And as, further, it is easily verified numerically that $B$ is finite when $M$ is zero, the series must stop at the term independent of $M$ and it follows that the proper integral function to fit to $B$ is a cubic in $M$.

Nothing in the above excludes the possibility of the occurrence of a fractional term in $B$, with a non-vanishing denominator, say of type $\frac{LM+N}{1+M^2}$, but it was felt that the simpler formula should be tried first, and should be adopted if the fit was at all reasonable.

14. To the values of $B$ calculated as in par. 11 was then fitted by least squares a formula

$$B = B_0 + B_1 M + B_2 M^2 + B_3 M^3 = \phi(M).$$

The values of the coefficients so obtained are shewn in the following table. For purposes of comparison the values of $-\mathscr{A}'$ and $\mathscr{B}'$ are given next to the corresponding $B_3$ and $B_2$ with which they should agree.

| $C/F$ | $B_0$ | $B_1$ | $B_2$ | $\mathscr{B}'$ | $B_3$ | $-\mathscr{A}'$ |
|---|---|---|---|---|---|---|
| 1·930943 | − ·6910 | − 3·5152 | 13·3978 | 12·3805 | − 9·5140 | − 9·4659 |
| ·961539 | − 1·2217 | ·4425 | 4·8790 | 4·8330 | − 4·2870 | − 4·2702 |
| 0 | − 1·4955 | 1·1825 | 1·6093 | 1·5310 | − 1·6097 | − 1·5600 |
| − ·961539 | − 1·7721 | ·1751 | 2·2923 | 2·3296 | − 1·0264 | − 1·0745 |
| − 1·930943 | − 1·8088 | − 3·2329 | 7·5686 | 6·8977 | − 2·7142 | − 2·6460 |

We note in passing that the agreement between $-\mathscr{A}'$ and $B_3$ is quite tolerable and between $B_2$ and $\mathscr{B}'$ is fair.

The values of $\phi(M)$ were calculated and compared with the values of $B$. The average divergences were found to be as follows:

| | | | |
|---|---|---|---|
| For $C/F =$ | 1·930943 | mean $\Delta B$ | $= ·0523$ |
| ,, $=$ | ·961539 | ,, | $= ·1632$ |
| ,, $=$ | 0 | ,, | $= ·2083$ |
| ,, $= -$ | ·961539 | ,, | $= ·0445$ |
| ,, $= -$ | 1·930943 | ,, | $= ·1873$ |

Thus the average $\Delta B$, i.e. divergence from the best fitting cubic of the $B$ calculated by two rays, is for all the five lenses

·1311.

For an emergence inclination of 10°, this will give a proportional error in the calculated longitudinal aberration of about ·004, or of 1 in 250. For an inclination of 5° the proportional error would be less than ·001. This accuracy is probably as much as is likely to be required by the designing optician. We may therefore accept the cubic in $M$ as giving a satisfactory fit for $B$.

15. The coefficients $B_0, B_1, B_2, B_3$ were then plotted to $C/F$. A casual inspection shewed that a parabola would in each case give a fair fit to the curve. A quadratic function of $C/F$ was therefore fitted by least squares and the final formula for $B$ then took the following form:

$$B = -1{\cdot}538273 + 1{\cdot}358249\,M + 1{\cdot}455563\,M^2 - 1{\cdot}561723\,M^3$$
$$+ \frac{C}{F}\{{\cdot}288802 - {\cdot}030931\,M + 1{\cdot}476767\,M^2 - 1{\cdot}747783\,M^3\}$$
$$+ \frac{C^2}{F^2}\{{\cdot}075455 - 1{\cdot}261500\,M + 2{\cdot}414432\,M^2 - 1{\cdot}218827\,M^3\} \quad ..(19).$$

The coefficients $B_0$, $B_1$, $B_2$, $B_3$ calculated from formula (19) are shewn in the table below.

Coefficients in $B = \phi(M)$ calculated from (19).

| $C/F$ | $B_0$ | $B_1$ | $B_2$ | $B_3$ |
|---|---|---|---|---|
| 1·930943 | − ·6993 | − 3·4050 | + 13·3094 | − 9·4810 |
| ·961539 | − 1·1908 | + ·1622 | + 5·1078 | − 4·3692 |
| 0 | − 1·5383 | + 1·3582 | + 1·4556 | − 1·5617 |
| − ·961539 | − 1·7462 | + ·2217 | + 2·2679 | − 1·0080 |
| − 1·930943 | − 1·8146 | − 3·2856 | + 7·6063 | − 2·7313 |

These coefficients agree only approximately with those given in the table of the last paragraph, and therefore it is to be expected that the agreement with the individual values of $B$ will not be so good.

To test the agreement of the final formula with these individual values, the value of $B$ given by the formula was calculated for all the individual cases and the comparison is given below. In the table below $F$ stands for the value of $B$ as given by the formula, $I$ for the individual value calculated directly from the two selected rays.

Fit of values of $B$.

| | $C/F = 1{\cdot}930943$. | | | $C/F = {\cdot}961539$. | | |
|---|---|---|---|---|---|---|
| $M$ | $F$ | $I$ | $F-I$ | $F$ | $I$ | $F-I$ |
| 3·0 | − 147·1176 | − 147·5307 | + ·4131 | − 72·7013 | − 71·6602 | − ·0411 |
| 2·0 | − 30·1199 | − 30·2700 | + ·1501 | − 15·3885 | − 15·2379 | − ·1506 |
| 1·5 | − 7·8591 | − 7·9095 | + ·0504 | − 4·2009 | − 4·1575 | − ·0434 |
| 0·5 | − ·2596 | − ·1662 | − ·0934 | − ·3789 | − ·2833 | − ·0956 |
| 0·1 | − ·9162 | − 1·0029 | + ·0867 | − 1·1279 | − 1·0125 | − ·1154 |
| 0 | − ·6993 | − ·7219 | + ·0226 | − 1·1908 | − 1·0854 | − ·1054 |
| − 0·1 | − ·2162 | − ·3065 | + ·0903 | − 1·1516 | − 1·1644 | + ·0128 |
| − 0·5 | 5·5157 | 5·7058 | − ·1901 | + ·5512 | + ·5695 | − ·0183 |
| − 1·0 | 25·4962 | 25·7508 | − ·2546 | + 8·1240 | + 6·9147 | + 1·2093 |
| − 2·0 | 135·1968 | 136·0338 | − ·8370 | + 53·8694 | + 51·8524 | + 2·0170 |

| $C/F=0$. | | | | $C/F=-\cdot961539$. | | |
|---|---|---|---|---|---|---|
| $M$ | $F$ | $I$ | $F-I$ | $F$ | $I$ | $F-I$ |
| 3·0 | −26·5300 | −26·8556 | +·3256 | − 7·8873 | − 8·3048 | +·4175 |
| 2·0 | − 5·4933 | − 5·6979 | +·2046 | − ·2957 | − ·4613 | +·1656 |
| 1·5 | − 1·4967 | − 1·6321 | +·1354 | + ·2869 | + ·0276 | +·2593 |
| 0·5 | − ·6905 | − ·6630 | −·0275 | − 1·1944 | − 1·2515 | +·0571 |
| 0·1 | − 1·3895 | − 1·2732 | −·1163 | − 1·7024 | − 1·7085 | +·0061 |
| 0 | − 1·5383 | − 1·3811 | −·1572 | − 1·7462 | − 1·7512 | +·0050 |
| −0·1 | − 1·6580 | − 1·3549 | −·3031 | − 1·7447 | − 1·7129 | −·0318 |
| −0·5 | − 1·6583 | − 1·5323 | −·1260 | − 1·1641 | − 1·2128 | +·0487 |
| −1·0 | + ·1208 | + ·1353 | −·0145 | + 1·3080 | + 1·4564 | −·1484 |
| −2·0 | +14·0613 | +15·5782 | −·5169 | +14·9463 | +15·2708 | −·3245 |

$C/F=-1\cdot930943$.

| $M$ | $F$ | $I$ | $F-I$ |
|---|---|---|---|
| 3·0 | −16·9596 | −16·8242 | −·1354 |
| 2·0 | + ·1891 | + ·8346 | −·6455 |
| 1·5 | + 1·1531 | + ·9474 | +·2057 |
| 0·5 | − 1·8972 | − 2·1772 | +·2800 |
| 0·1 | − 2·0698 | − 2·1359 | +·0661 |
| 0 | − 1·8146 | − 1·8413 | +·0267 |
| −0·1 | − 1·4072 | − 1·4089 | +·0017 |
| −0·5 | + 2·0712 | + 2·2093 | −·1381 |
| −1·0 | +11·8086 | +11·9238 | −·1152 |
| −2·0 | +57·0322 | +56·5384 | +·4938 |

Now in considering the effect of the divergences $F-I$ on the longitudinal aberration, it is important to remember that the range of $t'_4$ varies considerably with the magnification. If $y$ denote the distance at which the ray considered crosses the first principal plane, $y$ is the quantity which should be considered independent of $M$.

Now $$t_0=\frac{y}{1-\frac{1}{M}}=Mt'_4, \text{ so that } t'_4=\frac{y}{M-1}.$$

But the proportional error in computing $q$ due to a change $\Delta B$ in $B$ will, in general, be of the order $\Delta B t'^2_4$, i.e. $\Delta B y^2/(1-M)^2$. Hence in computing the proportional error for a ray crossing the first principal plane at a distance of ·1 from the axis, we find that this error will be (·01) [average value of $\Delta B/(1-M)^2$] and this will largely discount the importance of the comparatively large values of $F-I$ for the ends of the range of $M$.

From the above table we find the following for the mean numerical values of $\Delta B/(1-M)^2$ for the five lenses:

| $C/F$ | mean $[\Delta B/(1-M)^2]$ |
|---|---|
| 1·930943 | ·1274 |
| ·961539 | ·1510 |
| 0 | ·1606 |
| − ·961539 | ·1669 |
| −1·930943 | ·2877 |

giving a mean of ·1787 for all five lenses.

Hence the proportional average divergence in calculating $q$ for $y=\cdot1$ will be

·0018, that is, on the whole, the effect of calculating $B$ from the formula, instead of using the individual $B$'s, will be to vary the longitudinal aberration, on the average, by 1/500th part of itself.

No doubt a more complicated formula for the coefficients $B_0$, $B_1$, $B_2$, $B_3$ in terms of $C/F$ might have yielded a closer fit. In this connection we may remember that the coefficient $A$, for which we have the theoretical expression (10), is not expressible as a rational function of $C/F$, when the thickness of the lens is finite.

Having regard, however, to the approximate character of the whole investigation, it did not seem worth while to complicate the formulae.

16. Finally, having established the formula (13) where $A$ is given by the right-hand side of (10) and $B$ is given by (19), a number of isolated cases were taken in which the actual longitudinal aberration, computed trigonometrically, was compared with the longitudinal aberration, (1) as given by our formula, (2) as given by the formula $At_4^2$, corresponding to the usual first order aberration, and (3) as given by the formula $At'^2_4$ which is the first order aberration formula with the true slope replaced by the "Gaussian" slope. This will give us some measure of the extent to which the new formula takes account of the residual aberration over and above the first order term.

Comparison of true $q$ and $q$ calculated by empirical formula.

$\Delta_1 q = q$ (true) $- q$ (formula); $\Delta_{\#} q = q$ (true) $- At_4^2$; $\Delta_2 q = q$ (true) $- A(t'_4)^2$.

| $C/F$ | $M$ | $y$ | $q$ (true) | $q$ (formula) | $\Delta_1 q$ | $At_4^2$ | $\Delta_{\#} q$ | $A(t'_4)^2$ | $\Delta_2 q$ |
|---|---|---|---|---|---|---|---|---|---|
| 1·930943 | 3·0 | ·05 | ·163977 | ·163980 | − ·000003 | ·128688 | + ·035289 | ·148903 | + ·015074 |
| | ,, | ·15 | 7·874215 | 7·770459 | + ·103756 | ·063502 | + 7·810713 | 1·340123 | + 6·534092 |
| | 1·5 | ·05 | ·026420 | ·026338 | + ·000082 | ·022256 | + ·004164 | ·024268 | + ·002152 |
| | ,, | ·15 | ·759946 | ·746243 | + ·013703 | ·045664 | + ·714282 | ·218410 | + ·541536 |
| | 0·5 | ·05 | ·000387 | ·000387 | ·000000 | ·000389 | − ·000002 | ·000386 | + ·000001 |
| | ,, | ·15 | ·003530 | ·003560 | − ·000030 | ·003706 | − ·000176 | ·003477 | + ·000053 |
| | 0 | ·05 | ·006617 | ·006608 | + ·000009 | ·006734 | − ·000117 | ·006596 | + ·000021 |
| | ,, | ·15 | ·060427 | ·060314 | + ·000113 | ·072418 | − ·011991 | ·059365 | + ·001062 |
| | − 2·0 | ·05 | ·144238 | ·144372 | − ·000134 | ·166835 | − ·022597 | ·149794 | − ·005356 |
| | ,, | ·15 | 1·005306 | 1·007586 | − ·002280 | 3·368275 | − 2·362969 | 1·348143 | − ·342837 |
| 0 | 3·0 | ·05 | ·031123 | ·031123 | ·000000 | ·029713 | + ·001410 | ·030607 | + ·000516 |
| | ,, | ·15 | ·324391 | ·323783 | + ·000608 | ·206401 | + ·117990 | ·275465 | + ·048926 |
| | 1·5 | ·05 | ·008395 | ·008383 | + ·000012 | ·008006 | + ·000389 | ·008258 | + ·000137 |
| | ,, | ·15 | ·087119 | ·085889 | + ·001230 | ·055319 | + ·031800 | ·074320 | + ·012799 |
| | 0·5 | ·05 | ·003481 | ·003483 | − ·000002 | ·003501 | − ·000020 | ·003459 | + ·000022 |
| | ,, | ·15 | ·033096 | ·033191 | − ·000095 | ·035052 | − ·001956 | ·031128 | + ·001968 |
| | 0 | ·05 | ·003913 | ·003915 | − ·000002 | ·003928 | − ·000015 | ·003900 | + ·000013 |
| | ,, | ·15 | ·036211 | ·036358 | − ·000147 | ·037542 | − ·001331 | ·035100 | + ·001111 |
| | − 2·0 | ·05 | ·024970 | ·024943 | + ·000027 | ·025469 | − ·000499 | ·025041 | − ·000071 |
| | ,, | ·15 | ·217242 | ·217712 | − ·000470 | ·262437 | − ·045195 | ·225366 | − ·008124 |
| − 1·930943 | 3·0 | ·05 | ·010618 | ·010648 | − ·000030 | ·010359 | + ·000259 | ·010535 | + ·000083 |
| | ,, | ·15 | ·104441 | ·104818 | − ·000377 | ·080895 | + ·023546 | ·094819 | + ·009622 |
| | 1·5 | ·05 | ·001739 | ·001742 | − ·000003 | ·001742 | − ·000003 | ·001762 | − ·000023 |
| | ,, | ·15 | ·014558 | ·014364 | + ·000194 | ·014399 | + ·000159 | ·015854 | − ·001296 |
| | 0·5 | ·05 | ·013761 | ·013746 | + ·000015 | ·014070 | − ·000309 | ·013485 | + ·000276 |
| | ,, | ·15 | ·150678 | ·146353 | + ·004325 | ·205121 | − ·054443 | ·121364 | + ·029314 |
| | 0 | ·05 | ·023887 | ·023899 | − ·000012 | ·024717 | − ·000830 | ·023791 | + ·000096 |
| | ,, | ·15 | ·223242 | ·223230 | + ·000012 | ·316983 | − ·093741 | ·214116 | + ·009126 |
| | − 2·0 | ·05 | ·096777 | ·096767 | + ·000010 | ·104100 | − ·007323 | ·098300 | − ·001523 |
| | ,, | ·15 | ·775113 | ·774299 | + ·000814 | 1·481913 | − ·706800 | ·884699 | − ·109586 |

The values selected for comparison were $M = 3$, 1·5, ·5, 0 and $-2$ for each of the three lenses $C/F = \pm 1{\cdot}930943$, and $C/F = 0$. The inclinations corresponded to intercepts $y$ of ·05 and ·15 from the axis upon the first principal plane.

A mere casual inspection of the quantities $\Delta_1 q$, $\Delta_2 q$, $\Delta_3 q$ in the above table is sufficient to shew what a very great advance the empirical formula is on the first order aberration, especially where the divergence between the latter and the true values is considerable.

To form a numerical estimate of the improvements we find the mean numerical values of $\Delta_1 q$, $\Delta_2 q$, $\Delta_3 q$ in the above table.

$$\begin{aligned}
&\text{Mean } |\, \Delta_1 q \,| = {\cdot}004283 \\
&\text{Mean } |\, \Delta_2 q \,| = {\cdot}401344 = 93{\cdot}71 \text{ (mean } \Delta_1 q) \\
&\text{Mean } |\, \Delta_3 q \,| = {\cdot}255901 = 59{\cdot}75 \text{ (mean } \Delta_1 q).
\end{aligned}$$

The bulk of this, however, is contributed by an extreme case, namely $M = 3$, $y = 1{\cdot}5$ for the first lens, where the divergences of the true $q$ from $At_4^2$ and $A(t'_4)^2$ are of the order of several units and many times greater than $At_4^2$ or $A(t'_4)^2$ themselves, whereas the formula gives quite a decent approximation. If however we exclude this heavily weighted favourable case, we find

$$\begin{aligned}
&\text{Mean } \Delta_1 q = {\cdot}000853 \\
&\text{Mean } \Delta_2 q = {\cdot}145848 = 170{\cdot}98 \text{ (mean } \Delta_1 q) \\
&\text{Mean } \Delta_3 q = {\cdot}039411 = \phantom{1}46{\cdot}20 \text{ (mean } \Delta_1 q).
\end{aligned}$$

We may therefore say that the formula removes, on the average, 99·4 per cent. of the residual aberration left when the formula $At_4^2$ is used and 97·8 per cent. of the residual aberration left when the formula $A(t'_4)^2$ is used.

Incidentally it appears that a better average approximation is obtained from the first order formula if we use the "Gaussian" slope of the emergent ray, than if we use the true slope. The improvement due to the empirical formula is clearly very great and quite beyond what would be obtained by simply adding a second order aberration term.

This is very clearly shewn if we apply the formula to calculate $q$ for $M = 5$ and $y = {\cdot}15$ in the lens $C/F = 1{\cdot}930943$. In this case the true value of $q$ comes out to be $-19{\cdot}3836$; the critical value $\bar{a}_0$ has been passed and $q$, after passing through $\infty$, has now a large negative value; as we have seen, no formula obtained by adding aberrations of successive orders can then be valid. The first order formulae give $At_4^2 = +{\cdot}3448$, $A(t'_4)^2 = 4{\cdot}3342$, and are hopelessly out. Nevertheless, our empirical formula leads to $q = -19{\cdot}1263$ and thus, even in this extreme case, where the formula for $B$ has been carried well beyond the range of $M$ for which it was established, it determines the longitudinal aberration correctly within about $1\frac{1}{4}$ per cent. A similar calculation for $C/F = {\cdot}961539$, $M = 5$, $y = 1{\cdot}5$ leads to

$$\begin{aligned}
q \text{ (true)} &= 4{\cdot}9054 \qquad & At_4^2 &= {\cdot}4433 \\
q \text{ (formula)} &= 5{\cdot}0636 \qquad & At'^2_4 &= 2{\cdot}0812
\end{aligned}$$

which confirms the validity of the empirical formula.

So far, the formula has only been tested on the five lenses which were used in building it. The following examples are taken from two lenses *not* belonging to this set.

| | | | |
|---|---|---|---|
| $C/F = \cdot 484534$, $M = -1$, $y = \cdot 2$ | | | $\Delta_1 q = \quad \cdot 000248$ |
| $q$ (true) | $= \cdot 206998$ | $At_4^2 = \cdot 274343$ | $\Delta_2 q = -\cdot 067345$ |
| $q$ (formula) | $= \cdot 206750$ | $At'^2_4 = \cdot 212963$ | $\Delta_3 q = -\cdot 005965$ |
| $C/F = \cdot 690306$, $M = 0$, $y = \cdot 15$ | | | $\Delta_1 q = -\cdot 000108$ |
| $q$ (true) | $= \cdot 022998$ | $At_4^2 = \cdot 023968$ | $\Delta_2 q = -\cdot 000970$ |
| $q$ (formula) | $= \cdot 023106$ | $At'^2_4 = \cdot 022429$ | $\Delta_3 q = +\cdot 000569$ |
| | | $y = \cdot 225$ | |
| $q$ (true) | $= \cdot 053563$ | $At_4^2 = \cdot 059027$ | $\Delta_1 q = -\cdot 000466$ |
| $q$ (formula) | $= \cdot 054029$ | $At'^2_4 = \cdot 050465$ | $\Delta_2 q = -\cdot 005464$ |
| | | | $\Delta_3 q = +\cdot 003098$ |

These results confirm those already found and we may conclude that the formula is generally dependable for the range $C/F = -2$ to $C/F = +2$ and for a range of $M$ going up to quite high values both positive and negative. $C/F = \pm 2$ gives a meniscus lens of which Fig. 6 gives an accurate representation.

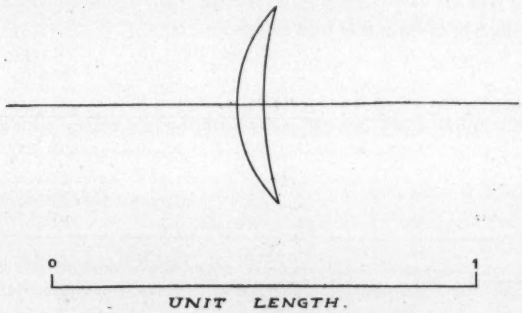



Fig. 6.

17. Certain interesting observations may be made concerning the values of the coefficients of formula (19). In the first place comparison of the values of $B_3$ (both the individual values given in par. 14 and the values derived from the quadratic formula in $C/F$ and given in par. 15) with the values of $A_4$ in the development

$$A = A_0 + A_1 M + A_2 M^2 + A_3 M^3 + A_4 M^4$$

leads to a practical identity of the coefficients $B_3$ and $-A_4$. The coefficients in the development of $A$ are given below and the coefficients $B_3$ from the tables in pars. 14 and 15 are added for comparison.

The reason for the approximate identity of $B_3$ and $-A_4$ is that, as explained in par. 13, $B_3$ for $C/F = +K$ should be sensibly equal to $-\mathscr{A}'$, i.e. to $-\mathscr{A}$, where $\mathscr{A}$ is the value of $A$ for $C/F = -K$ and $M = 0$, i.e. to $A_0$ for $C/F = -K$, which is identical with $A_4$ for $C/F = +K$.

Coefficients in $A$.

| $C/F$ | $A_0$ | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $B_3$ (individual) | $B_3$ (formula) |
|---|---|---|---|---|---|---|---|
| 1·930943 | +2·638431 | −14·922744 | +31·379419 | −28·624400 | +9·516270 | −9·5140 | −9·4810 |
| ·961539 | 1·072270 | − 5·366086 | 11·752569 | −11·742604 | 4·272189 | −4·2870 | −4·3692 |
| 0 | 1·559999 | − 4·230616 | 5·330015 | − 4·230616 | 1·559999 | −1·6097 | −1·5617 |
| − ·961539 | 4·272189 | −11·742604 | 11·752569 | − 5·366086 | 1·072270 | −1·0264 | −1·0080 |
| −1·930943 | 9·516270 | −28·624400 | 31·379419 | −14·922744 | 2·638431 | −2·7142 | −2·7313 |

It may be that $B_3$ should be taken exactly equal to $-A_4$ and, in fact, if the formula is to hold for *all* values of $M$ however large, this must be done.

In this case it is clear that $B_3$ can never be exactly fitted by a quadratic in $C/F$, since $A_4$ as given by equation (10) cannot be so expressed. But if we regard the formula for $B$ as purely empirical, the constants are best left as they are for practical approximation.

Again, if we look at (19) and take the sum of the coefficients, we obtain the following results:

Sum of coefficients in term independent of $C/F = -\cdot286184$
" " " multiplied by $C/F = -\cdot013145$
" " " " $(C/F)^2 = +\cdot009560$

which suggests that $M-1$ is an approximate factor of $B$.

In other words, $B$ will be small in the neighbourhood of $M=1$ for all values of $C/F$. Examination of equations (10) and (11) shews that the same holds good of $A$.

Rewriting $B$ in the form

$$\begin{aligned} B = &-\cdot286184 - \cdot013145\frac{C}{F} + \cdot009560\left(\frac{C}{F}\right)^2 \\ &+ (1-M)[1\cdot561723M^2 + \cdot106160M - 1\cdot252089] \\ &+ (1-M)\frac{C}{F}[1\cdot747783M^2 + \cdot271016M + \cdot301947] \\ &+ (1-M)\left(\frac{C}{F}\right)^2[1\cdot218827M^2 - 1\cdot195605M + \cdot065895], \end{aligned}$$

the terms in the first line must necessarily be part of those in $B$ due to the thickness of the lens, for, in the ideal *thin* lens, the aberration is accurately nil when $M=1$.

18. More work will be required before these empirical formulae can be made available for lenses of any thickness and any refractive index. It will be necessary to repeat the whole of this work, (1) with a set of ideal *thin* lenses of the same curvature, to see how the coefficients vary with the thickness; (2) with a set of lenses of a different refractive index, to find how they are affected by change in the refractive index; (3) with negative lenses.

Further, before the method can be applied to calculate the complete path of any emergent ray and thus to investigate the aberrations off the axis, and also for dealing in a similar fashion with aberrations through a system of two or more

lenses, it will be necessary to obtain empirical formulae on the same lines giving the slope $\tan \alpha_4$ of the emergent ray in terms of $\tan \alpha_0$. The authors are at present engaged on working out such formulae and it is hoped to publish the results at no distant date.

Meanwhile enough has been done to shew that a formula of the type $q = \dfrac{At'^3_4}{1 + Bt'^3_4}$ provides a rapid and practically accurate method of obtaining the longitudinal spherical aberration in a lens, without the use of the laborious trigonometrical calculations. A set of double entry tables for $A$ and $B$ in terms of $C/F$ and $M$, which could be easily computed from formulae of the type (11) and (19), would undoubtedly save the designing optician much time and labour, and it is hoped that the construction of such tables may be taken in hand by those interested in the progress of optical design in this country.

## DISCUSSION

Mr S. D. Chalmers said he was interested to see how the authors had made use of values obtained for first order aberrations in their empirical formula for higher order aberrations. He himself had calculated the second order aberrations directly some years ago, and had found in the first case he tested that the third order aberrations were producing as much effect as the second order aberrations. He concluded that the best procedure was to deduce modified coefficients for the second order aberrations using the results of exact calculations and the theoretical first order coefficient. The authors' formulae seemed to give closer agreement than one would have expected.

Mr T. Smith congratulated the authors on a very arduous and valuable piece of work. The chief novelty in their paper was not in his opinion the appearance of variable terms in the denominator—for it was in the denominator that aberration terms naturally first appeared, transference to the numerator resulting from expansion—but in the discovery that such close approximation over a considerable range of values of $C/F$ and $M$ was obtainable from an expression with one $t^2$ term in the numerator and one in the denominator. He regretted that he had been unable to shew, as he intended, how the errors outstanding with the authors' formulae compared with some derived theoretically. That the expansion of the exact expression for the longitudinal aberration could not always be permitted was evident from the appearance of the sine of the emergent inclination in the denominator. Suppose that a ray passes through a system of $n$ surfaces of which $\lambda$ is a typical example, separating media of refractive indices $\mu_{\lambda-1}$ and $\mu_\lambda$. Let $h_\lambda$ and $h'_\lambda$ be the perpendiculars to the incident and refracted rays at this surface from the centre of curvature. Then $\mu_{\lambda-1} h_\lambda = \mu_\lambda h'_\lambda$, so that

$$\begin{aligned} \mu_0 h_1 - \mu_n h'_n &= \sum_{\lambda=1}^{n-1} \mu_\lambda (h'_\lambda - h_{\lambda+1}) \\ &= -\sum_{\lambda=1}^{n-1} \mu_\lambda a_\lambda \sin \psi_\lambda, \end{aligned}$$

where $a_\lambda$ is the distance between the centres of curvature of surfaces $\lambda$ and $\lambda + 1$ and $\psi_\lambda$ is the inclination to the axis of the ray in its passage between these two surfaces. If the incident ray strikes the axis in the object space at a distance $x$ from the first surface, and the emergent ray meets it at a distance $x'$ from the last surface,

$$h_1 = -x \sin \psi_0, \quad h'_n = -x' \sin \psi_n.$$

Thus

$$\mu_n x' \sin \psi_n = \mu_0 x \sin \psi_0 - \sum_1^{n-1} \mu_\lambda a_\lambda \sin \psi_\lambda.$$

The expression for the longitudinal aberration is found by dividing throughout by $\mu_n \sin \psi_n$ and subtracting the value of $x'$ for paraxial rays. Since the equation may be written in the form

$$x' \frac{\mu_n \sin \psi_n}{b} = x \frac{\mu_0 \sin \psi_0}{b} - \sum_1^{n-1} a_\lambda \frac{\mu_\lambda \sin \psi_\lambda}{b},$$

where $b$ may be given any convenient value, such as $\mu_0 \sin \psi_0$ or the length of the perpendicular to the incident ray from the first nodal point, or from the vertex of the first surface, it follows that the denominator may be regarded as a corrected value of the sine ratio. Expansion of the expression for the longitudinal aberration in an ordinary power series is only permissible if the variation of the sine ratio is small compared with its paraxial value. This limitation does not apply to expressions for the latitudinal aberrations.

To find a critical test for the formula it was necessary to examine the numerical results it gave in the neighbourhood of the zeroes and stationary points for the first order aberrations. At two of the zeroes the authors had themselves noted that the accuracy fell off. With the particular thickness they had employed the other zeroes were imaginary for the range of values of $C/F$ within which high accuracy was claimed. The result of a trial calculation* inside that range gave lower accuracy than with most of the illustrative results obtained by the authors, but still sufficiently accurate for practical applications. It should however be noted that the range of $C/F$ from $-3$ to $+3$ was too restricted. For instance lenses would undoubtedly be used with the rays at the first surface passing through the aplanatic points, and at the second surface through the centre of curvature. This required a curvature of six for the first surface, a much higher value than the authors had considered. The formula should be tested up to and beyond this stage, making the range of values of $M$ a function of $C/F$.

Any proposals to extend the use of a formula of this type should be received with caution. It would undoubtedly break down if applied to cemented objectives where the aberration changes sign for an intermediate zone of the lens. The same would probably be true for single lenses if the thickness were introduced as an independent variable, a generalisation essential for extensive applications of the formula.

* The numerical values quoted at the meeting were incorrect, the actual spherical aberration being given too high a value owing to the substitution of the focal distance of the object space for that of the image space. The value of $B$ was also wrong owing to the occurrence of incorrect signs in the MS of the paper.

He had little doubt that to obtain a really satisfactory general formula it would be necessary to adopt the form

$$\frac{\alpha S^2 + \beta S^4}{1 + \gamma S^2 + \delta S^4},$$

where $S$ is the sine of the inclination of the incident ray, or some transformation of this. In numerous cases $\delta$ could probably be neglected. If such a formula were adopted it would be almost certainly found that values of $\alpha$, $\beta$, $\gamma$, $\delta$ obtained on a theoretical basis rather than an empirical one determined by the method of least squares gave satisfactory results. The adoption of theoretical values would afford the advantage that the coefficients required to express $\gamma$ and $\delta$ in terms of the magnification are the same as those involved in $\alpha$ and $\beta$. For example, in a notation already explained*, $\alpha$ is a multiple of

$$A - (4B + 1)m + 2(3C + \varpi)m^2 - (4B' + 1)m^3 + A'm^4$$

and $\gamma$ on this theoretical basis is a multiple of

$$B - (3C + \varpi)m + (3B' + 1)m^2 - A'm^3$$

and a somewhat similar relation connects $\beta$ and $\delta$. He considered that the difficulty of calculating coefficients for second order aberrations had been much exaggerated.

Dr R. S. Clay thought the authors deserved great praise for the enormous amount of laborious work connected with the computation involved. He asked whether it was possible, by using the results obtained from tracing one ray trigonometrically, to obtain a sufficiently close approximation to the value of $B$ for use in the rough computation of the aberrations for other rays.

The authors, in a written reply to Mr Smith's criticism, admit that in dealing with the longitudinal aberration the possibility of a breakdown in the convergency of the series may occur, where such a possibility may be absent if the latitudinal aberration is considered. The longitudinal aberration was chosen as it is the one which is almost always available in trigonometrical calculations and in any such case the lateral aberration is calculable from it except in the special case of the longitudinal aberration being infinite.

It is also admitted that the investigation of the validity of the formula for the case of a lens for which $C/F = 3$ should have been made before reading the paper. The case cited by Mr Smith at the meeting and referred to in his footnote on p. 88 has been worked out in detail and the following results obtained:

| | |
|---|---|
| Longitudinal aberration obtained trigonometrically | 0·001465 |
| Aberration calculated by formula .. .. | 0·001481 |
| Aberration by first order formula .. .. | 0·001650. |

It will be seen from this that the formula value is only slightly more than 1 per cent. in error as against 12 per cent. error in the first order value.

The range of $C/F$ from $-3$ to $+3$ covers the bulk of the type of lenses for which this formula is designed and it was not thought necessary in dealing with an empirical formula to consider extreme cases. For the special case of the lens that forms an image at the aplanatic point of the first surface and at the centre of the

* *Trans. Opt. Soc.* 18 (1917) 176.

second the formula is exact. The details of the particular case with this worked out with the same kind of glass as before are:

$$r_1 = 0{\cdot}1666667$$
$$r_3 = 0{\cdot}2138158$$
$$\text{Thickness} = 0{\cdot}0625$$
$$C/F = 5{\cdot}338461$$
$$M = 1/n = 0{\cdot}657895.$$

A mere glance at the shape of such a lens when drawn to scale will shew that an assumption that the lens is thin is quite beside the mark and that the thin lens formula (6) for the calculation of $A$ would be unjustifiable. The inaccuracy of (6) was recognised all through our work and the values of $A$ where given as *calculated* values are derived from (10). Using the formula for the above data we obtain $A = 0$ which is exactly correct as the image point chosen is a true aplanatic point for the lens.

It should be noticed that in the bulk of cases that have to be considered the value of $A$ calculated from (6) is not very different from the true value as calculated from (10). For this particular case, which is extreme, the value of $A$ obtained by (6) is 0·0527, and the longitudinal aberration for a ray inclined on emergence at about 45° with the axis is 0·0128, one-eightieth part of the focal length.

Mr Smith is of course right in stating that the formula will break down in the case of a lens in which the aberration changes sign for an intermediate zone of the lens. The formula can only change sign once, and through infinity, as steeper rays are chosen. That point had already been considered and modifications of the formula (1) for greater accuracy and (2) for the purpose of transferring aberrations from one lens to another have been in existence for some time and will shortly be published. These formulae will probably cover the case that Mr Smith has in mind.

The form in which the complete longitudinal aberration can be expressed is without doubt something of the nature of a fraction consisting of one infinite series divided by another. It is in the manner in which approximations to these two series are made that Mr Smith and the authors are in disagreement.

Taking the general case we can write it:

$$q = \frac{\alpha t^2 + \beta t^4 + \gamma t^6 + \ldots\ldots}{1 + B' t^2 + \gamma' t^4 + \ldots\ldots}.$$

A similar expression is also possible with the series expanded in powers of the sine instead of the tangent but the coefficients will be different except for $\alpha$. When $t$ is sufficiently small the fraction can immediately be expanded as a single power series and the first, second, third, etc. orders of aberration will appear in this series. Approximations to $q$ which will contain aberrations of any required order can be made by stopping short the numerator and denominator at suitable points. The first order aberration is given obviously by putting $q = \frac{\alpha t^2}{1}$ and the second by writing $q = \frac{\alpha t^2 + \beta t^4}{1 + B' t^2}$. The inclusion of the third order requires the employment

of $\gamma t^6$ in the numerator and $\gamma' t^4$ in the denominator. On the other hand the expression $q = \dfrac{\alpha t^2}{1 + B' t^2}$ will agree with the expanded power series only as far as the first order aberration and is, in general, no improvement on the ordinary first order formula $q = \alpha t^2$; at the same time the second order form $q = \dfrac{\alpha t^2 + \beta t^4}{1 + B' t^2}$ can also be written

$$q = \frac{\alpha t^2 \left(1 + \dfrac{\beta}{\alpha} t^2\right)}{1 + B' t^2},$$

which can be written as approximately equal to

$$\frac{\alpha t^2}{1 + \left(B' - \dfrac{\beta}{\alpha}\right) t^2}.$$

This is really the form which we have used in our paper. This expression when expanded as a power series contains correctly the second order term for the lens.

Mr Smith seems to have assumed that the authors' formula is obtained by neglecting terms in $t^4$ in both numerator and denominator, which is not the case. They are led to this conclusion by his statement above in which he gives a formula for $B$ in terms of the first order coefficients. That formula appears to be correct for $B'$ but is most certainly incorrect for $B$ as the following examples will shew. In these calculations the values $B_0$, $B_1$, $B_2$ in the formula

$$B = B_0 + B_1 M + B_2 M^2 + B_3 M^3$$

have been worked out both by the authors' formula contained in equation (19) and also calculated from Mr Smith's expression:

$$B - (3C + \varpi)\, m + (3B' + 1)\, m^2 - A' m^3$$

| | $C/F = 1{\cdot}93$ | | $C/F = 0$ | | $C/F = -1{\cdot}93$ | |
|---|---|---|---|---|---|---|
| Coefft. | Mr Smith | Actual | Mr Smith | Actual | Mr Smith | Actual |
| $B_2$ | 21·593 | 13·309 | 3·298 | 1·456 | 11·317 | 7·606 |
| $B_1$ | −15·690 | −3·405 | −2·665 | 1·358 | 15·690 | −3·286 |
| $B_0$ | 4·606 | − ·699 | 1·933 | −1·538 | 8·031 | −1·814 |

Mr Smith's absolute values obtained above cannot be insisted on as he does not state what is the multiple required but it is quite clear from the above comparison that Mr Smith's cubic in the magnification is most certainly not a multiple of ours.

What the authors have in mind is that the longitudinal aberration which, for rays of sufficiently small inclination with the axis, may certainly be expanded in the form

$$\alpha t^2 + b t^4 + c t^6 + \ldots.$$

can be expressed as a fraction of the form

$$\frac{\alpha t^2 + \beta t^4 + \gamma t^6 + \ldots\ldots}{1 + B t^2 + \gamma' t^4 + \ldots\ldots},$$

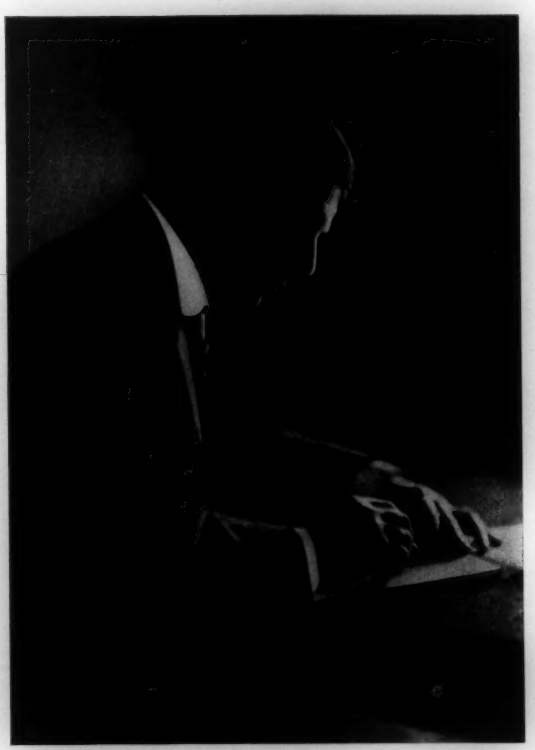
where $\beta, \gamma$, etc. in the numerator are absolutely arbitrary and $\beta', \gamma'$, etc. are found by dividing $\alpha t^2 + \beta t^4 + \gamma t^6$ by $\alpha t^2 + bt^4 + ct^6 + \ldots$.

For the particular form in which the authors' formula has been put, $\beta, \gamma$, etc. have been chosen zero and the denominator series has been stopped at the second term. This formula is therefore correct as far as the second order aberration and as the numbers shew its numerical accuracy certainly goes beyond that point in general.

The formula

$$\frac{\alpha t^2}{1 + \left(B' - \frac{\beta}{\alpha}\right) t^2}$$

breaks down for a case in which the first order aberration is zero but higher orders are not zero. This has been recognised and a formula of the form $\frac{At^2 + Et^4}{1 + Bt^2}$ is under consideration for the purpose of dealing with such cases. The coefficient $B$ is in this case slightly different from the previous $B$.



WILLIAM JAMES McCULLOCH ETTLES

President, Optical Society, 1914–16

# OBITUARY NOTICE

## WILLIAM JAMES McCULLOCH ETTLES, M.D.

It is with great regret that we have to record the loss of William James McCulloch Ettles, M.D., M.S., F.R.C.S.E., of 114 *a*, Harley Street, London, W., who died after a severe surgical operation on July 19th, 1918. He was for two years in succession President of the Optical Society, and took a very keen interest in the discussion of optical questions, particularly those relating to Ophthalmology.

He was the son of the late John Ettles, of Elgin, and was born at Brighton in 1869. Educated at Aberdeen, Edinburgh and Paris Universities, he had an unusually long list of academical honours, and held, or had held, many ophthalmological positions of importance.

He was a Member of the Court of Assistants of the Worshipful Company of Gold and Silver Wyre Drawers, a Liveryman of the Worshipful Company of Spectaclemakers, Consulting Oculist to the Eastern Public Dispensary and London Spectacle Mission; Treasurer (late President) of the Hunterian Society; Knight Hospitaller of the Order of St John the Baptist of Spain; Foreign Member of the Spanish Red Cross; Honorary Fellow and Medallist (1st Class) Italian Physical Academy; Vice-President and Hon. Medical Officer, London Morayshire Club; late Pathologist, Royal Eye Hospital; Lecturer on Physiology of Vision, Northampton Institute, and Society of Illuminating Engineers; Consulting Oculist, Car and General Insurance Corporation, etc.; Special Lecturer, London College of Ambulance; Clinical Assistant at Royal Westminster Ophthalmic Hospital.

Many opticians will entertain kindly recollections of him as an examiner. For a short period he was an examiner in visual optics for the Spectaclemakers' Company and afterwards for many years until his untimely death was on the examining staff of the British Optical Association, where he conducted a special oral examination in which the general resourcefulness of the candidate was tested.

He was the author of many ophthalmological treatises, amongst others being: Numerous articles in various medical journals on diseases of the eye; "The Renaissance of Ophthalmology during the Hunterian Era," *Hunt. Oration*, 1908; "Buphthalmus," *Trans. Hunt. Soc.*, 1893; "Optical Principles of the Ophthalmometer, with Description of New Instrument," *Trans. Optical Convention*, 1905; "Pathology and Treatment of Ocular Complications of Gonorrhoeal Infections," *Lancet*, 1907; "The History of Optical Signalling in Peace and War," *Trans. Opt. Soc.*, 1916.

One of his special subjects was that of Ophthalmometry, or Keratometry, in

which he was one of our foremost authorities. He was the inventor of many ophthalmic instruments, notably the one in which his name is associated with that of the late Mr Curtis in the "Curtis-Ettles Ophthalmometer."

He was a keen sportsman, and in his Sussex home at West Wittering was well known for the strong interest he took in outdoor recreations, such as yachting, etc. A man of versatile attainments, as a violoncellist a musician of ability, an indefatigable and artistic collector, a skilled connoisseur in ancient firearms and weapons, all he undertook was characterised by a thoroughness and patience not often met with.

He was a man of strong convictions, and perhaps, above all other ophthalmic surgeons, managed to work on exceedingly good terms with all opticians with whom he came in contact. It is very much to be doubted whether any ophthalmic surgeon was held in such high esteem by the opticians of this country, who all sincerely deplore his loss. No man more than himself foresaw that it was to the combined interest of the patient, the surgeon, and the optician, that the ophthalmic surgeon and optician should work together in friendly harmony.

He leaves a widow and two sons, the elder of whom, Lt. Ian Ettles, is in the R.F.A.; the younger, Mr Donald Ettles, is a student at Marlborough College.

J. H. S.

# ABSTRACTS

"Measuring the Focal Length of a Photographic Lens." By T. SMITH, B.A., *Proc. Phys. Soc.* 27 (1915), 171.

The method described is applicable to combinations consisting of two positive or converging components. The lens is set up in the camera or on an optical bench and focussed on a distant object lying on the axis of the lens. The front component of the lens is then removed without disturbing the rear component, and the distance through which the ground glass or examining microscope has to be moved back to bring the same into focus is measured and denoted by $d$. The front component is now replaced, and the lens reversed, and the same operation repeated, the other component being removed and the displacement of focus $d'$ measured. The focal length of the combination is $\sqrt{dd'}$. These two measurements also give the ratio of the focal lengths of the front and back components. Their absolute values may be found by taking a supplementary observation in which the separation of the components is varied.

This method of measuring focal lengths is to be preferred to others which demand the observation or measurement of images off the axis, as it will be less affected by aberrations.

"On tracing Rays through an Optical System." By T. SMITH, B.A., *Proc. Phys. Soc.* 27 (1915), 502.

An algebraic method of tracing any ray through an optical system is developed. The ray is identified for a preliminary calculation by the cosine of its inclination to the axis and the length of the perpendicular to the ray from the centre of curvature of a refracting surface. The preliminary calculation yields the length of the optical path between successive surfaces and a quantity $K$ which becomes equal to $\kappa$, the Gaussian power, for paraxial rays. The quantities $K$ for the various surfaces are combined according to the same laws as the $\kappa$'s, except that the length of ray intercepted between the surfaces is substituted for their axial separation. The Gaussian laws connecting the incident and emergent directions of the ray and the points in which it meets the first and last surfaces may then be generalised by substituting the sines of angles for the angles themselves and for any function of the $\kappa$'s the corresponding function of the $K$'s.

When the ray lies in an axial plane the $K$ functions determine the positions of conjugate secondary focal lines.

"Notes on the Calculation of 'thin' Objectives." By T. SMITH, B.A., *Proc. Phys. Soc.* 27 (1915), 485.

The first order aberrations of any centered system of lenses may be expressed in terms of six coefficients which are independent of the position of the object or of the stop, just as the Gaussian functions are independent of the same quantities. The formulae for the calculation of these six aberration coefficients are assumed and it is pointed out that when all the thicknesses and separations are neglected three identical relations exist

between them. The formulae are suitably transformed for the calculation of a doublet with an air gap, and numerical examples are worked out. The effect of bending a thin combination of any number of component lenses is worked out. Formulae are found for the calculation of cemented triple objectives, the aberration coefficients of which are expressed in terms of those of a cemented double objective made of the same kinds of glass. Examples of the calculations of triple objectives of both types are given, and diagrams are added shewing the relation between refractive index and dispersion required to make the simultaneous correction of spherical aberration and coma possible in a cemented doublet.

T. S.